

CONFORMAL SYMMETRY BREAKING OPERATORS FOR DIFFERENTIAL FORMS ON SPHERES

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ABSTRACT. We give a complete classification of conformally covariant differential operators between the spaces of i -forms on the sphere S^n and j -forms on the totally geodesic hypersphere S^{n-1} . Moreover, we find explicit formulæ for these new matrix-valued operators in the flat coordinates in terms of basic operators in differential geometry and classical orthogonal polynomials. We also establish matrix-valued factorization identities among all possible combinations of conformally covariant differential operators. The main machinery of the proof is the “F-method” based on the “algebraic Fourier transform of Verma modules” (Kobayashi–Pevzner [Selecta Math. 2016]) and its extension to matrix-valued case developed here. A short summary of the main results was announced in [C. R. Acad. Sci. Paris, 2016].

Key words and phrases: *Symmetry breaking operators, branching law, F-method, conformal geometry, Verma module, Lorentz group.*

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1. INTRODUCTION

Let (X, g) be a pseudo-Riemannian manifold. Suppose that a Lie group G acts conformally on X . This means that there exists a positive-valued function $\Omega \in C^\infty(G \times X)$ (*conformal factor*) such that

$$L_h^* g_{h \cdot x} = \Omega(h, x)^2 g_x \quad \text{for all } h \in G, x \in X,$$

where we write $L_h: X \rightarrow X, x \mapsto h \cdot x$ for the action of G on X . When X is orientable, we define a locally constant function $\text{or}: G \times X \rightarrow \{\pm 1\}$ by $\text{or}(h)(x) = 1$ if $(L_h)_*x: T_x X \rightarrow T_{L_h x} X$ is orientation-preserving, and $= -1$ if it is orientation-reversing.

Since Ω satisfies a cocycle condition, we can form a family of representations $\varpi_{u, \delta}^{(i)}$ of G with parameters $u \in \mathbb{C}$ and $\delta \in \mathbb{Z}/2\mathbb{Z}$ on the space $\mathcal{E}^i(X)$ of i -forms on X ($0 \leq i \leq \dim X$) defined by

$$(1.1) \quad \varpi_{u, \delta}^{(i)}(h)\alpha := \text{or}(h)^\delta \Omega(h^{-1}, \cdot)^u L_{h^{-1}}^* \alpha, \quad (h \in G).$$

The representation $\varpi_{u, \delta}^{(i)}$ of the conformal group G on $\mathcal{E}^i(X)$ will be simply denoted by $\mathcal{E}^i(X)_{u, \delta}$, and referred to as conformal representations on i -forms.

Suppose that Y is an orientable submanifold such that g is nondegenerate on the tangent space $T_y Y$ for all $y \in Y$ (this holds automatically if g is positive definite). Then Y is endowed with a pseudo-Riemannian structure $g|_Y$, and we can define in a similar way a family of representations $\varpi_{v, \varepsilon}^{(j)}$ on $\mathcal{E}^j(Y)$ ($v \in \mathbb{C}, \varepsilon \in \mathbb{Z}/2\mathbb{Z}, 0 \leq j \leq \dim Y$) of the group

$$G' := \{h \in G : h \cdot Y = Y\}$$

which acts conformally on $(Y, g|_Y)$.

The object of our study is differential operators $\mathcal{D}^{i \rightarrow j}: \mathcal{E}^i(X) \rightarrow \mathcal{E}^j(Y)$ that intertwine the two representations $\varpi_{u, \delta}^{(i)}|_{G'}$ and $\varpi_{v, \varepsilon}^{(j)}$ of G' . Here $\varpi_{u, \delta}^{(i)}|_{G'}$ stands for the restriction of the G -representation $\varpi_{u, \delta}^{(i)}$ to the subgroup G' . We say that such $\mathcal{D}^{i \rightarrow j}$ is a *differential symmetry breaking operator* and denote by $\text{Diff}_{G'}(\mathcal{E}^i(X)_{u, \delta}, \mathcal{E}^j(Y)_{v, \varepsilon})$ the space of differential symmetry breaking operators.

We address the following problems:

Problem A. Find a necessary and sufficient condition on 6-tuple $(i, j, u, v, \delta, \varepsilon)$ such that there exist nontrivial differential symmetry breaking operators. More precisely, determine the dimension of $\text{Diff}_{G'}(\mathcal{E}^i(X)_{u,\delta}, \mathcal{E}^j(Y)_{v,\varepsilon})$.

Problem B. Construct explicitly a basis of $\text{Diff}_{G'}(\mathcal{E}^i(X)_{u,\delta}, \mathcal{E}^j(Y)_{v,\varepsilon})$.

In the case where $X = Y$, $G = G'$, and $i = j = 0$, a classical prototype of such operators is a second order differential operator called the Yamabe operator

$$\Delta + \frac{n-2}{4(n-1)}\kappa \in \text{Diff}_G(\mathcal{E}^0(X)_{\frac{n}{2}-1,\delta}, \mathcal{E}^0(X)_{\frac{n}{2}+1,\delta}),$$

where n is the dimension of the manifold X , Δ is the Laplace–Beltrami operator, and κ is the scalar curvature, see [18], for instance. Conformally equivariant differential operators of higher order are also known: the Paneitz operator (fourth order) [24], which appears in four dimensional supergravity [9], or more generally, the so-called GJMS operators ([10]) are such operators. Analogous differential operators on forms ($i = j$ case) were studied by Branson [4]. The exterior derivative d and the codifferential d^* also give examples of conformally covariant operators on forms, namely, $j = i+1$ and $i-1$, respectively, with appropriate choice of $(u, v, \delta, \varepsilon)$. Maxwell’s equations in four dimension can be expressed in terms of conformally covariant operators on forms.

Let us consider the more general case where $Y \neq X$ and $G' \neq G$. An obvious example of symmetry breaking operators is the restriction operator Rest_Y which belongs to $\text{Diff}_{G'}(\mathcal{E}^i(X)_{u,\delta}, \mathcal{E}^i(Y)_{v,\varepsilon})$ if $u = v$ and $\delta \equiv \varepsilon \equiv 0 \pmod{2}$. Another elementary example is $\text{Rest}_Y \circ \iota_{N_Y(X)} \in \text{Diff}_{G'}(\mathcal{E}^i(X)_{u,\delta}, \mathcal{E}^{i-1}(Y)_{v,\varepsilon})$ with $v = u + 1$ and $\delta \equiv \varepsilon \equiv 1 \pmod{2}$ where $\iota_{N_Y(X)}$ denotes the interior multiplication by the normal vector field to Y in X when Y is of codimension one in X (see Proposition 8.12).

In the model space $(X, Y) = (S^n, S^{n-1})$, the pair (G, G') of conformal groups amounts to $(O(n+1, 1), O(n, 1))$ modulo center (see Lemma 11.1), and Problems A and B have been recently solved for $i = j = 0$ by Juhl [11]. See also [15] and [19] for different approaches, *i.e.*, by the residue calculus and the F-method, respectively. The classification of nonlocal symmetry breaking operators for $i = j = 0$ has been also accomplished recently in [22]. On the other hand, the case $n = 2$ with $(i, j) = (1, 0)$ was studied in [16] with emphasis on the relation to the Rankin–Cohen brackets [5, 7, 25].

This work gives a complete solution to Problems A and B for all i and j in the model space $(X, Y) = (S^n, S^{n-1})$: we classify all differential symmetry breaking operators from i -forms on S^n to j -forms on S^{n-1} for all i and j . We also find closed formulæ for these new operators in all the cases.

The key machinery of the proof is the F-method which has been recently introduced in [14] by the first author. See also [15, 19, 20] for detailed account and some

applications. The idea of the F-method is based on the “algebraic Fourier transform of Verma modules”. We shall develop an extension of the method to the matrix-valued case in Chapter 3.

Let us state our main results. Here is a complete solution to Problem A for the model space $(X, Y) = (S^n, S^{n-1})$ ($n \geq 3$).

Theorem 1.1. *Let $n \geq 3$. Suppose $0 \leq i \leq n$, $0 \leq j \leq n-1$, $u, v \in \mathbb{C}$, $\delta, \varepsilon \in \mathbb{Z}/2\mathbb{Z}$. Then the following three conditions on 6-tuple $(i, j, u, v, \delta, \varepsilon)$ are equivalent:*

- (i) $\text{Diff}_{O(n,1)}(\mathcal{E}^i(S^n)_{u,\delta}, \mathcal{E}^j(S^{n-1})_{v,\varepsilon}) \neq \{0\}$,
- (ii) $\dim_{\mathbb{C}} \text{Diff}_{O(n,1)}(\mathcal{E}^i(S^n)_{u,\delta}, \mathcal{E}^j(S^{n-1})_{v,\varepsilon}) = 1$,
- (iii) *One of the following twelve conditions holds.*

Case (I). $j = i - 2$, $2 \leq i \leq n - 1$, $(u, v) = (n - 2i, n - 2i + 3)$, $\delta \equiv \varepsilon \equiv 1 \pmod{2}$.

Case (I'). $(i, j) = (n, n - 2)$, $u \in -n - \mathbb{N}$, $v = 3 - n$, $\delta \equiv \varepsilon \equiv u + n + 1 \pmod{2}$.

Case (II). $j = i - 1$, $1 \leq i \leq n$, $v - u \in \mathbb{N}_+$, $\delta \equiv \varepsilon \equiv v - u \pmod{2}$.

Case (III). $j = i$, $0 \leq i \leq n - 1$, $v - u \in \mathbb{N}$, $\delta \equiv \varepsilon \equiv v - u \pmod{2}$.

Case (IV). $j = i + 1$, $1 \leq i \leq n - 2$, $(u, v) = (0, 0)$, $\delta \equiv \varepsilon \equiv 0 \pmod{2}$.

Case (IV'). $(i, j) = (0, 1)$, $u \in -\mathbb{N}$, $v = 0$, $\delta \equiv \varepsilon \equiv u \pmod{2}$.

Case (*I). $j = n - i + 1$, $2 \leq i \leq n - 1$, $u = n - 2i$, $v = 0$, $\delta \equiv 1$, $\varepsilon \equiv 0 \pmod{2}$.

Case (*I'). $(i, j) = (n, 1)$, $u \in -n - \mathbb{N}$, $v = 0$, $\delta \equiv \varepsilon + 1 \equiv u + n + 1 \pmod{2}$.

Case (*II). $j = n - i$, $1 \leq i \leq n$, $v - u + n - 2i \in \mathbb{N}$, $\delta \equiv \varepsilon + 1 \equiv v - u + n + 1 \pmod{2}$.

Case (*III). $j = n - i - 1$, $0 \leq i \leq n - 1$, $v - u + n - 2i - 1 \in \mathbb{N}$, $\delta \equiv \varepsilon + 1 \equiv v - u + n + 1 \pmod{2}$.

Case (*IV). $j = n - i - 2$, $1 \leq i \leq n - 2$, $(u, v) = (0, 2i - n + 3)$, $\delta \equiv 0$, $\varepsilon \equiv 1 \pmod{2}$.

Case (*IV'). $(i, j) = (0, n - 2)$, $u \in -\mathbb{N}$, $v = 3 - n$, $\delta \equiv \varepsilon + 1 \equiv u \pmod{2}$.

We shall give a proof of Theorem 1.1 in Section 11.3.

There are dualities in the twelve cases in Theorem 1.1. To be precise, we set

$$\tilde{i} := n - i, \tilde{j} := n - j - 1, \tilde{u} := u + 2i - n, \tilde{v} := v + 2j - n + 1, \tilde{\delta} \equiv \delta + 1, \tilde{\varepsilon} \equiv \varepsilon + 1 \pmod{2}.$$

Then it follows from the Hodge duality for symmetry breaking operators (Theorem 8.8, see also Section 11.1) that $(i, j, u, v, \delta, \varepsilon) \mapsto (\tilde{i}, \tilde{j}, \tilde{u}, \tilde{v}, \tilde{\delta}, \tilde{\varepsilon})$ gives rise to the duality

of parameters

$$(I) \iff (IV), \quad (I') \iff (IV'), \quad (II) \iff (III),$$

and $(i, j, u, v, \delta, \varepsilon) \mapsto (i, \tilde{j}, u, \tilde{v}, \delta, \tilde{\varepsilon})$ gives rise to another duality of parameters

$$(P) \iff (*P) \quad \text{for } P = I, I', II, III, IV, IV'.$$

Differential symmetry breaking operators for the latter half, *i.e.*, Cases $(*I)$ – $(*IV')$, are given as the composition of the Hodge star operator $*_{\mathbb{R}^{n-1}}$ and the corresponding symmetry breaking operators for the first half, *i.e.*, Cases (I) – (IV') .

The equivalence (i) \Leftrightarrow (ii) in Theorem 1.1 asserts that differential symmetry breaking operators, if exist, are unique up to scalar multiplication for all the parameters $(i, j, u, v, \delta, \varepsilon)$ if $n \geq 3$. This should be in contrast to the $n = 2$ case, where the multiplicity jumps at countably many places to two (*cf.* [21, Sect. 9]).

The standard sphere S^n is a conformal compactification of the flat Riemannian manifold \mathbb{R}^n . Using the stereographic projection $p : S^n \rightarrow \mathbb{R}^n \cup \{\infty\}$, we give closed formulæ of differential symmetry breaking operators in flat coordinates in Cases (I), (I'), (II), (III), (IV), and (IV'), see Theorems 1.8, 1.5, 1.6 and 1.7, respectively. Change of coordinates in symmetry breaking operators from \mathbb{R}^n to the conformal compactification S^n is given by the “twisted pull-back” of the stereographic projection in Section 11.5. In order to explain the explicit formulæ of the symmetry breaking operators, in the flat coordinates, we fix some notations for basic operators.

Suppose that a manifold X is endowed with a pseudo-Riemannian structure g of signature (p, q) and an orientation. Then, the metric tensor g induces a volume form vol_X , and a pseudo-Riemannian structure on the cotangent bundle $T^\vee X$, or more generally on the exterior power bundles $\bigwedge^i(T^\vee X)$. The codifferential $d^* : \mathcal{E}^i(X) \rightarrow \mathcal{E}^{i-1}(X)$ is the formal adjoint of the differential (exterior derivative) d in the sense that

$$\int_X g_x(\alpha, d\beta) \text{vol}_X(x) = \int_X g_x(d^*\alpha, \beta) \text{vol}_X(x)$$

for all $\alpha \in \mathcal{E}^i(X)$ and $\beta \in \mathcal{E}^{i-1}(X)$. Interior multiplication ι_Z of an i -form ω by a vector field Z is defined by

$$(\iota_Z \omega)(Z_1, \dots, Z_{i-1}) := \omega(Z, Z_1, \dots, Z_{i-1}).$$

For $\ell \in \mathbb{N}$ and $\mu \in \mathbb{C}$, we denote by $\tilde{C}_\ell^\mu(t)$ the Gegenbauer polynomial which is renormalized in a way that $\tilde{C}_\ell^\mu \not\equiv 0$ for any $\mu \in \mathbb{C}$ (see (14.3) in Appendix). Then

$$(I_\ell \tilde{C}_\ell^\mu)(x, y) := x^{\frac{\ell}{2}} \tilde{C}_\ell^\mu\left(\frac{y}{\sqrt{x}}\right)$$

is a polynomial of two variables x and y . We replace formally x by $-\Delta_{\mathbb{R}^{n-1}} = -\sum_{j=1}^{n-1} \frac{\partial^2}{\partial x_j^2}$ and y by $\frac{\partial}{\partial x_n}$, and define a family of scalar-valued differential operators

on \mathbb{R}^n of order ℓ

$$(1.2) \quad \mathcal{D}_\ell^\mu := (I_\ell \tilde{C}_\ell^\mu) \left(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n} \right).$$

For instance, $\mathcal{D}_0^\mu = 1$, $\mathcal{D}_1^\mu = 2\frac{\partial}{\partial x_n}$, $\mathcal{D}_2^\mu = \Delta_{\mathbb{R}^{n-1}} + 2(\mu + 1)\frac{\partial^2}{\partial x_n^2}$, $\mathcal{D}_3^\mu = 2\Delta_{\mathbb{R}^{n-1}}\frac{\partial}{\partial x_n} + \frac{4}{3}(\mu + 2)\frac{\partial^3}{\partial x_n^3}$, etc. We regard $\mathcal{D}_\ell^\mu \equiv 0$ for negative integer ℓ .

For $\mu \in \mathbb{C}$ and $a \in \mathbb{N}$, we set

$$(1.3) \quad \gamma(\mu, a) := \frac{\Gamma(\mu + 1 + [\frac{a}{2}])}{\Gamma(\mu + [\frac{a+1}{2}])} = \begin{cases} 1 & \text{if } a \text{ is odd,} \\ \mu + \frac{a}{2} & \text{if } a \text{ is even.} \end{cases}$$

For $1 \leq i \leq n$, we introduce a family of linear maps $\mathcal{D}_{u,a}^{i \rightarrow i-1} : \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^{i-1}(\mathbb{R}^{n-1})$ with parameters $u \in \mathbb{C}$ and $a \in \mathbb{N}$ by

$$(1.4) \quad \mathcal{D}_{u,a}^{i \rightarrow i-1} := \text{Rest}_{x_n=0} \circ \left(-\mathcal{D}_{a-2}^{\mu+1} d_{\mathbb{R}^n}^* d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} - \gamma(\mu, a) \mathcal{D}_{a-1}^{\mu+1} d_{\mathbb{R}^n}^* + \frac{1}{2}(u + 2i - n) \mathcal{D}_a^\mu \iota_{\frac{\partial}{\partial x_n}} \right)$$

$$(1.5) \quad = \text{Rest}_{x_n=0} \circ \left(-\mathcal{D}_{a-2}^{\mu+1} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n} + \frac{1}{2}(u + 2i - n + a) \mathcal{D}_a^\mu \iota_{\frac{\partial}{\partial x_n}} \right)$$

$$- \gamma(\mu - \frac{1}{2}, a) d_{\mathbb{R}^{n-1}}^* \circ \text{Rest}_{x_n=0} \circ \mathcal{D}_{a-1}^\mu,$$

where $\mu := u + i - \frac{1}{2}(n-1)$ and $\iota_{\frac{\partial}{\partial x_n}}$ stands for the interior multiplication by the vector field $\frac{\partial}{\partial x_n}$. Then, $\mathcal{D}_{u,a}^{i \rightarrow i-1}$ is a matrix-valued homogeneous differential operator of order a . See Definition 3.2 for the precise meaning of “differential operators between two manifolds”. The proof of the second equality (1.5) will be given in Proposition 9.9.

Example 1.2. Here are some few examples of the operators $\mathcal{D}_{u,a}^{i \rightarrow i-1}$ for $i = 1, n$ or $a = 0, 1$, and 2:

$$\begin{aligned} \mathcal{D}_{u,a}^{1 \rightarrow 0} &= \text{Rest}_{x_n=0} \circ \left(-\gamma(u - \frac{n-3}{2}, a) \mathcal{D}_{a-1}^{u-\frac{n-5}{2}} d_{\mathbb{R}^n}^* + \frac{1}{2}(u + 2 - n) \mathcal{D}_a^{u-\frac{n-3}{2}} \iota_{\frac{\partial}{\partial x_n}} \right), \\ \mathcal{D}_{u,a}^{n \rightarrow n-1} &= \frac{1}{2}(u + n + a) \text{Rest}_{x_n=0} \circ \mathcal{D}_a^{u+\frac{n+1}{2}} \iota_{\frac{\partial}{\partial x_n}}, \\ \mathcal{D}_{u,0}^{i \rightarrow i-1} &= \frac{1}{2}(u + 2i - n) \text{Rest}_{x_n=0} \circ \iota_{\frac{\partial}{\partial x_n}}, \\ \mathcal{D}_{u,1}^{i \rightarrow i-1} &= \text{Rest}_{x_n=0} \circ \left(-d_{\mathbb{R}^n}^* + (u + 2i - n) \frac{\partial}{\partial x_n} \iota_{\frac{\partial}{\partial x_n}} \right), \\ \mathcal{D}_{u,2}^{i \rightarrow i-1} &= \text{Rest}_{x_n=0} \circ D, \end{aligned}$$

where $D = \left(-d_{\mathbb{R}^n}^* d_{\mathbb{R}^n}^* + \frac{1}{2}(u + 2i - n) \left(\Delta_{\mathbb{R}^{n-1}} + (n + 2i + 5) \frac{\partial^2}{\partial x_n^2} \right) \right) \iota_{\frac{\partial}{\partial x_n}} - 2\gamma \frac{\partial}{\partial x_n} d_{\mathbb{R}^n}^*$.

For $0 \leq i \leq n-1$, we introduce another family of linear maps $\mathcal{D}_{u,a}^{i \rightarrow i} : \mathcal{E}^i(\mathbb{R}^n) \longrightarrow \mathcal{E}^i(\mathbb{R}^{n-1})$ with parameters $u \in \mathbb{C}$ and $a \in \mathbb{N}$ by

(1.6)

$$\mathcal{D}_{u,a}^{i \rightarrow i} := \text{Rest}_{x_n=0} \circ \left(\mathcal{D}_{a-2}^{\mu+1} d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* - \gamma\left(\mu - \frac{1}{2}, a\right) \mathcal{D}_{a-1}^{\mu} d_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}} + \frac{1}{2}(u+a) \mathcal{D}_a^{\mu} \right)$$

(1.7)

$$= -d_{\mathbb{R}^{n-1}}^* d_{\mathbb{R}^{n-1}} \circ \text{Rest}_{x_n=0} \circ \mathcal{D}_{a-2}^{\mu+1} + \text{Rest}_{x_n=0} \circ \left(\gamma(\mu, a) \mathcal{D}_{a-1}^{\mu+1} \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n} + \frac{u}{2} \mathcal{D}_a^{\mu} \right),$$

where $\mu = u + i - \frac{n-1}{2}$ as before. Then $\mathcal{D}_{u,a}^{i \rightarrow i}$ is a matrix-valued homogeneous differential operator of order a . The second equality (1.7) and an alternative definition of $\mathcal{D}_{u,a}^{i \rightarrow i}$ by means of the Hodge star operators

$$(1.8) \quad \mathcal{D}_{u,a}^{i \rightarrow i} := (-1)^{n-1} *_{\mathbb{R}^{n-1}} \circ \mathcal{D}_{u-n+2i,a}^{n-i \rightarrow n-i-1} \circ (*_{\mathbb{R}^n})^{-1}$$

will be proved in Proposition 10.3.

Example 1.3. Here are some few examples of the operators $\mathcal{D}_{u,a}^{i \rightarrow i}$ for $i = 0, n-1$ or $a = 0, 1$, and 2.

$$\mathcal{D}_{u,a}^{0 \rightarrow 0} = \frac{u+a}{2} \text{Rest}_{x_n=0} \circ \mathcal{D}_a^{u-\frac{n-1}{2}}.$$

$$\mathcal{D}_{u,a}^{n-1 \rightarrow n-1} = -d_{\mathbb{R}^{n-1}}^* d_{\mathbb{R}^{n-1}} \text{Rest}_{x_n=0} \circ \mathcal{D}_{a-2}^{\mu+1} + \frac{u}{2} \text{Rest}_{x_n=0} \circ \mathcal{D}_a^{\mu}.$$

$$\mathcal{D}_{u,0}^{i \rightarrow i} = \frac{u}{2} \text{Rest}_{x_n=0}.$$

$$\mathcal{D}_{u,1}^{i \rightarrow i} = \text{Rest}_{x_n=0} \circ \left(-d_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}} + (u+1) \frac{\partial}{\partial x_n} \right).$$

$$\mathcal{D}_{u,2}^{i \rightarrow i} = \text{Rest}_{x_n=0} \circ \left(d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* + \left(\frac{n}{2} + 1 \right) \Delta_{\mathbb{R}^n} + \left(u + i - \frac{n}{2} + 1 \right) \left((n+2) \frac{\partial^2}{\partial x_n^2} + 2 \frac{\partial}{\partial x_n} d_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}} \right) \right).$$

These differential operators are generically nonzero, however, they may vanish in specific cases. To be precise, we prove in Section 9.3:

Proposition 1.4. Suppose $u \in \mathbb{C}$ and $a \in \mathbb{N}$.

- (1) Let $1 \leq i \leq n$. Then $\mathcal{D}_{u,a}^{i \rightarrow i-1}$ vanishes if and only if $(u, a) = (n-2i, 0)$ or $(u, i) = (-n-a, n)$.
- (2) Let $0 \leq i \leq n-1$. Then $\mathcal{D}_{u,a}^{i \rightarrow i}$ vanishes if and only if $(u, a) = (0, 0)$ or $(u, i) = (-a, 0)$.

In order to obtain nonzero operators for all the parameters (i, a, u) , we renormalize $\mathcal{D}_{u,a}^{i \rightarrow i-1}$ and $\mathcal{D}_{u,a}^{i \rightarrow i}$, respectively, by

$$(1.9) \quad \tilde{\mathcal{D}}_{u,a}^{i \rightarrow i-1} := \begin{cases} \text{Rest}_{x_n=0} \circ \iota_{\frac{\partial}{\partial x_n}} & \text{if } a = 0, \\ \text{Rest}_{x_n=0} \circ \mathcal{D}_a^{u+\frac{n+1}{2}} \circ \iota_{\frac{\partial}{\partial x_n}} & \text{if } i = n, \\ \mathcal{D}_{u,a}^{i \rightarrow i-1} & \text{otherwise.} \end{cases}$$

$$(1.10) \quad \tilde{\mathcal{D}}_{u,a}^{i \rightarrow i} := \begin{cases} \text{Rest}_{x_n=0} & \text{if } a = 0, \\ \text{Rest}_{x_n=0} \circ \mathcal{D}_a^{u-\frac{n-1}{2}} & \text{if } i = 0, \\ \mathcal{D}_{u,a}^{i \rightarrow i} & \text{otherwise.} \end{cases}$$

Clearly, these operators are well-defined because the formulæ on the right-hand sides coincide in the overlapping cases such as $a = 0$ and $i = n$.

We are now ready to give a solution to Problem B when $j = i - 1$ and i .

Theorem 1.5 ($j = i - 1$). *Let $1 \leq i \leq n$. Suppose $(u, v) \in \mathbb{C}^2$ and $(\delta, \varepsilon) \in (\mathbb{Z}/2\mathbb{Z})^2$ satisfy $v - u \in \mathbb{N}_+$ and $\delta \equiv \varepsilon \equiv v - u \pmod{2}$. We set*

$$a := v - u - 1 \in \mathbb{N}.$$

- (1) *The differential operator $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i-1}$ extends to the conformal compactification S^n of \mathbb{R}^n , and induces a nontrivial $O(n, 1)$ -homomorphism $\mathcal{E}^i(S^n)_{u,\delta} \rightarrow \mathcal{E}^{i-1}(S^{n-1})_{v,\varepsilon}$, to be denoted by the same letter $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i-1}$.*
- (2) *Any $O(n, 1)$ -equivariant differential operator from $\mathcal{E}^i(S^n)_{u,\delta}$ to $\mathcal{E}^{i-1}(S^{n-1})_{v,\varepsilon}$ is proportional to $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i-1}$.*

Theorem 1.6 ($j = i$). *Let $0 \leq i \leq n - 1$. Suppose $(u, v) \in \mathbb{C}^2$ and $(\delta, \varepsilon) \in (\mathbb{Z}/2\mathbb{Z})^2$ satisfy $v - u \in \mathbb{N}$ and $\delta \equiv \varepsilon \equiv v - u \pmod{2}$. We set*

$$a := v - u \in \mathbb{N}.$$

- (1) *The differential operator $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i}$ extends to S^n , and induces a nontrivial $O(n, 1)$ -homomorphism $\mathcal{E}^i(S^n)_{u,\delta} \rightarrow \mathcal{E}^i(S^{n-1})_{v,\varepsilon}$, to be denoted by the same letter $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i}$.*
- (2) *Any $O(n, 1)$ -equivariant differential operator from $\mathcal{E}^i(S^n)_{u,\delta}$ to $\mathcal{E}^i(S^{n-1})_{v,\varepsilon}$ is proportional to $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i}$.*

In contrast to the above cases where $j = i - 1$ or i , we prove that differential symmetry breaking operators of higher order are rare for $j \notin \{i-1, i\}$. Let us describe all of them. For $j = i + 1$, a family of differential operators $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i+1}: \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^{i+1}(\mathbb{R}^{n-1})$ are defined by

$$(1.11) \quad \tilde{\mathcal{D}}_{u,a}^{i \rightarrow i+1} := \text{Rest}_{x_n=0} \circ \mathcal{D}_{-u}^{u-\frac{n-1}{2}} \circ d_{\mathbb{R}^n}$$

but only when $a = 1 - u$ with additional constraints $u = 0$ ($1 \leq i \leq n - 2$) in Case (IV) in Theorem 1.1; $u \in -\mathbb{N}$ ($i = 0$) in Case (IV'). We note $\tilde{\mathcal{D}}_{0,1}^{i \rightarrow i+1} = \text{Rest}_{x_n=0} \circ d_{\mathbb{R}^n}$ and $\tilde{\mathcal{D}}_{1-a,a}^{0 \rightarrow 1} = d \circ \tilde{\mathcal{D}}_{1-a,a-1}^{0 \rightarrow 0}$ (Theorem 13.18 (6)).

For $j = i - 2$, a family of differential operators $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i-2}: \mathcal{E}^i(\mathbb{R}^n) \longrightarrow \mathcal{E}^{i-2}(\mathbb{R}^{n-1})$ are defined by

$$(1.12) \quad \tilde{\mathcal{D}}_{u,a}^{i \rightarrow i-2} := \text{Rest}_{x_n=0} \circ \mathcal{D}_{-u+n-2i}^{u+\frac{n+1}{2}} \circ \iota_{\frac{\partial}{\partial x_n}} \circ d_{\mathbb{R}^n}^*,$$

but only when $a = 1 + n - 2i - u$ with additional constraints $u = n - 2i$ ($2 \leq i \leq n - 1$) in Case (I), $u \in \{-n, -n - 1, -n - 2, \dots\}$ ($i = n$) in Case (I'). We note $\tilde{\mathcal{D}}_{n-2i,1}^{i \rightarrow i-2} = \text{Rest}_{x_n=0} \circ \iota_{\frac{\partial}{\partial x_n}} \circ d_{\mathbb{R}^n}^*$ and $\tilde{\mathcal{D}}_{1-n-a,a}^{n \rightarrow n-2} = -d^* \circ \tilde{\mathcal{D}}_{1-n-a,a-1}^{n \rightarrow n-1}$ (see Theorem 13.18 (8)).

Then the solution to Problem B in the remaining cases, *i.e.*, $j \in \{i + 1, i - 2\}$ is stated as follows:

Theorem 1.7 ($j = i + 1$). *Let $0 \leq i \leq n - 2$. Suppose $(i, i + 1, u, v, \delta, \varepsilon)$ belongs to Case (IV) or (IV') in Theorem 1.1. In particular, $\delta \equiv \varepsilon \pmod{2}$, $u \in -\mathbb{N}$ and $v = 0$. We set*

$$a := v - u + 1 = 1 - u \in \mathbb{N}_+.$$

- (1) *The differential operator $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i+1}$ extends to the conformal compactification S^n , and induces a nontrivial $O(n, 1)$ -homomorphism $\mathcal{E}^i(S^n)_{u,\delta} \longrightarrow \mathcal{E}^{i+1}(S^{n-1})_{0,\delta}$, to be denoted by the same letter $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i+1}$.*
- (2) *Case (IV): Suppose $1 \leq i \leq n - 2$. Then any $O(n, 1)$ -equivariant differential operator $\mathcal{E}^i(S^n)_{0,0} \longrightarrow \mathcal{E}^{i+1}(S^{n-1})_{0,0}$ is proportional to $\tilde{\mathcal{D}}_{0,1}^{i \rightarrow i+1} = \text{Rest}_{S^{n-1}} \circ d_{S^n}$.*
- (3) *Case (IV'): Suppose $i = 0$. Then any $O(n, 1)$ -equivariant differential operator $\mathcal{E}^0(S^n)_{u,\delta} \longrightarrow \mathcal{E}^1(S^{n-1})_{0,\delta}$ ($u \in -\mathbb{N}$, $\delta \equiv u \pmod{2}$) is proportional to $\tilde{\mathcal{D}}_{u,1-u}^{0 \rightarrow 1}$.*

Theorem 1.8 ($j = i - 2$). *Let $2 \leq i \leq n$. Suppose $(i, i - 2, u, v, \delta, \varepsilon)$ belongs to Case (I) or (I') in Theorem 1.1. In particular, $\delta \equiv \varepsilon \pmod{2}$, $u \in -n - \mathbb{N}$ and $v = n - 2i + 3$. We set*

$$a := v - u - 2 = n - 2i + 1 - u \in \mathbb{N}_+.$$

- (1) *The differential operator $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i-2}$ extends to S^n , and induces a nontrivial $O(n, 1)$ -homomorphism $\mathcal{E}^i(S^n)_{u,\delta} \longrightarrow \mathcal{E}^{i-2}(S^{n-1})_{n-2i+3,\delta}$, to be denoted by the same letter $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i-2}$.*
- (2) *Case (I): Suppose $2 \leq i \leq n - 1$. Then any $O(n, 1)$ -equivariant differential operator $\mathcal{E}^i(S^n)_{n-2i,1} \longrightarrow \mathcal{E}^{i-2}(S^{n-1})_{n-2i+3,1}$ is proportional to $\tilde{\mathcal{D}}_{n-2i,1}^{i \rightarrow i-2} = \text{Rest}_{S^{n-1}} \circ \iota_{N_{S^{n-1}}(S^n)} \circ d_{S^n}^*$.*

- (3) Case (I'): Suppose $i = n$. Then any $O(n, 1)$ -equivariant differential operator $\mathcal{E}^n(S^n)_{u,\delta} \rightarrow \mathcal{E}^{n-2}(S^{n-1})_{3-n,\delta}$ ($u \in -n - \mathbb{N}, \delta \equiv u + n + 1 \pmod{2}$) is proportional to $\tilde{\mathcal{D}}_{u,1-u-n}^{n \rightarrow n-2}$.

Thus Problems A and B have been settled for the pair $(X, Y) = (S^n, S^{n-1})$.

Finally, we discuss matrix-valued functional identities (*factorization theorems*) arising from compositions of conformally equivariant operators. They are formulated as follows. Suppose that $T_X: \mathcal{E}^{i'}(X) \rightarrow \mathcal{E}^i(X)$ or $T_Y: \mathcal{E}^j(Y) \rightarrow \mathcal{E}^{j'}(Y)$ is a conformally equivariant operator for forms. Then the composition $T_Y \circ D$ or $D \circ T_X$ of a symmetry breaking operator $D = D_{X \rightarrow Y}: \mathcal{E}^i(X) \rightarrow \mathcal{E}^j(Y)$ with T_X or T_Y is again a symmetry breaking operator:

$$\begin{array}{ccc} \mathcal{E}^i(X)_{u,\delta} & \xrightarrow{D_{X \rightarrow Y}} & \mathcal{E}^j(Y)_{v,\varepsilon} \\ T_X \uparrow & \searrow & \downarrow T_Y \\ \mathcal{E}^{i'}(X)_{u',\delta'} & & \mathcal{E}^{j'}(Y)_{v',\varepsilon'} \end{array}$$

In the setting where $X = S^n$ (or $Y = S^{n-1}$, respectively), conformally covariant differential operators $T_X: \mathcal{E}^{i'}(X) \rightarrow \mathcal{E}^i(X)$ (or $T_Y: \mathcal{E}^j(Y) \rightarrow \mathcal{E}^{j'}(Y)$, respectively) are classified in Theorem 12.1. This case (*i.e.* $X = X$ or $Y = Y$) is much easier than the case $Y \subsetneq X$ treated in Theorem 1.1 for symmetry breaking operators. For the proof, we again use the F-method in a self-contained manner, although classical results of algebraic representation theory (*e.g.* [2]) could be used to simplify the proof. Thus we see that T_X (or T_Y , respectively) is proportional to d, d^* , Branson's operators $\mathcal{T}_{2\ell}^{(i)}$ (or $\mathcal{T}_{2\ell}^{(j)}$, respectively) of order 2ℓ (see (12.1)), or the composition of these operators with the Hodge star operator. On the other hand, the general multiplicity-freeness theorem (see Theorem 1.1) guarantees that such compositions must be proportional to the operators that we classified in Theorems 1.5, 1.6, 1.7 and 1.8.

In Chapter 13, we give a complete list of factorization identities with explicit proportionality constants for all possible cases. We illustrate the new factorization identities by taking T_X or T_Y to be Branson's operators $\mathcal{T}_{2\ell}^{(i)}$ or $\mathcal{T}_{2\ell}^{(j)}$ as follows. For $\ell \in \mathbb{N}_+$ and $a \in \mathbb{N}$, we define a positive number $K_{\ell,a}$ by

$$(1.13) \quad K_{\ell,a} := \prod_{k=1}^{\ell} \left(\left\lfloor \frac{a}{2} \right\rfloor + k \right).$$

Theorem 1.9 (See Theorem 13.1). *Suppose $0 \leq i \leq n$, $a \in \mathbb{N}$ and $\ell \in \mathbb{N}_+$. Then*

$$\begin{aligned} (1) \quad \mathcal{D}_{\frac{n}{2}-i+\ell,a}^{i \rightarrow i-1} \circ \mathcal{T}_{2\ell}^{(i)} &= - \left(\frac{n}{2} - i - \ell \right) K_{\ell,a} \mathcal{D}_{\frac{n}{2}-i-\ell,a+2\ell}^{i \rightarrow i-1} \quad \text{if } i \neq 0. \\ (2) \quad \mathcal{D}_{\frac{n}{2}-i+\ell,a}^{i \rightarrow i} \circ \mathcal{T}_{2\ell}^{(i)} &= - \left(\frac{n}{2} - i + \ell \right) K_{\ell,a} \mathcal{D}_{\frac{n}{2}-i-\ell,a+2\ell}^{i \rightarrow i} \quad \text{if } i \neq n. \end{aligned}$$

Theorem 1.10 (See Theorem 13.2). *Suppose $0 \leq i \leq n$, $a \in \mathbb{N}$ and $\ell \in \mathbb{N}_+$. We set $u := \frac{n-1}{2} - i - \ell - a$. Then*

$$\begin{aligned} (1) \quad \mathcal{T}_{2\ell}'^{(i-1)} \circ \mathcal{D}_{u,a}^{i \rightarrow i-1} &= - \left(\frac{n+1}{2} - i + \ell \right) K_{\ell,a} \mathcal{D}_{u,a+2\ell}^{i \rightarrow i-1} \quad \text{if } i \neq 0. \\ (2) \quad \mathcal{T}_{2\ell}'^{(i)} \circ \mathcal{D}_{u,a}^{i \rightarrow i} &= - \left(\frac{n-1}{2} - i - \ell \right) K_{\ell,a} \mathcal{D}_{u,a+2\ell}^{i \rightarrow i} \quad \text{if } i \neq n. \end{aligned}$$

The scalar case ($i = 0$) in Theorem 1.9 (2) and 1.10 (2) was studied in [11, 19], and was extended to all the symmetry breaking operators (including nonlocal ones) in [22]. The other matrix-valued factorization identities are given in Theorems 13.3 and 13.4, see also Theorems 13.15, 13.16, and 13.18 for the factorization identities of renormalized symmetry breaking operators. We also analyze when the proportionality constant vanishes.

Finally, let us mention analogous results for the connected groups, other real forms in pseudo-Riemannian geometry, and branching problems for Verma modules. Throughout the paper, we study Problems A and B in full detail for the whole group of conformal diffeomorphisms of S^n that preserves S^{n-1} , which is a disconnected group. Then results for the connected group $SO_0(n, 1)$, or equivalently, for conformal vector fields on S^n along the submanifold S^{n-1} , can be extracted from our main results for the disconnected group $O(n, 1)$, see Theorem 2.10.

Branching problems for (generalized) Verma modules for $(\mathfrak{g}, \mathfrak{g}') = (\mathfrak{o}(n+2, \mathbb{C}), \mathfrak{o}(n+1, \mathbb{C}))$ are the algebraic counterpart of Problems A and B for $(X, Y) = (S^n, S^{n-1})$ by a general duality theorem [19, 20] that gives a one-to-one correspondence between differential symmetry breaking operators and \mathfrak{g}' -homomorphisms for the restriction of Verma modules of \mathfrak{g} . Branching laws for Verma modules are discussed in Section 2.6.

Our results can be also extended to the non-Riemannian setting $S^{p,q} \supset S^{p-1,q}$ for the pair $(G, G') = (O(p+1, q), O(p, q))$ of conformal groups, for which the $(i, j) = (0, 0)$ case was studied in [19].

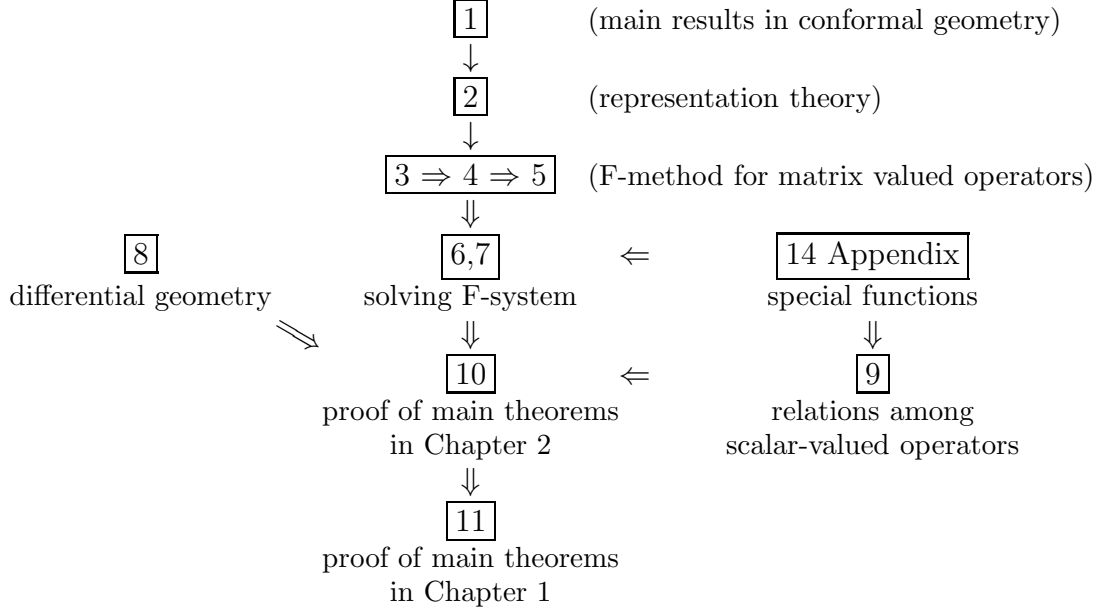
The main results were announced in [17].

Notation: $\mathbb{N} := \{0, 1, 2, \dots\}$, $\mathbb{N}_+ := \{1, 2, \dots\}$.

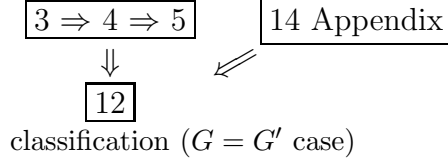
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The relation between chapters is illustrated by the following figures. Here, “ \Rightarrow ” means a strong relation (*e.g.* logical dependency), and “ \rightarrow ” means a weak relation (*e.g.* setup or definition).

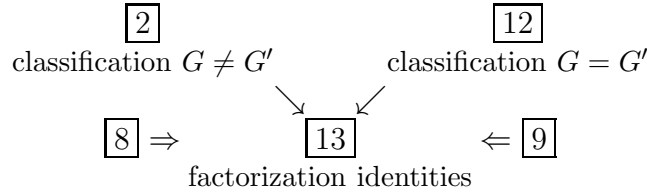
- Classification of differential symmetry breaking operators



- A baby case ($G = G'$) in Chapter 12 could be read independently:



- Factorization identities



2. SYMMETRY BREAKING OPERATORS AND PRINCIPAL SERIES REPRESENTATIONS OF $G = O(n+1, 1)$

The conformal compactification S^n of \mathbb{R}^n may be thought of as the real flag variety of the indefinite orthogonal group $G = O(n+1, 1)$, and the twisted action $\varpi_{u,\delta}^{(i)}$ of G on $\mathcal{E}^i(S^n)$ is a special case of the principal series representations of G . In this chapter, we reformulate the solutions to Problems A and B for $(X, Y) = (S^n, S^{n-1})$ given in Theorem 1.1 and Theorems 1.5-1.8, respectively, in Introduction in terms of symmetry breaking operators for principal series representations when $(G, G') = (O(n+1, 1), O(n, 1))$ in Theorems 2.7 and 2.8, respectively.

Some important properties (duality theorem of symmetry breaking operators, reducible places) of the principal series representations of $G = O(n+1, 1)$ are also discussed in this chapter.

2.1. Principal series representations of $G = O(n+1, 1)$. We set up notations for the group $O(n+1, 1)$ and its parabolically induced representations. Let $Q_{n+1,1}$ be the standard quadratic form of signature $(n+1, 1)$ on \mathbb{R}^{n+2} defined by

$$Q_{n+1,1}(x) := x_0^2 + x_1^2 + \cdots + x_n^2 - x_{n+1}^2 \quad \text{for } x = (x_0, x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+2},$$

and we realize the Lorentz group $O(n+1, 1)$ as

$$G := O(n+1, 1) = \{g \in GL(n+2, \mathbb{R}) : Q_{n+1,1}(gx) = Q_{n+1,1}(x) \text{ for all } x \in \mathbb{R}^{n+2}\}.$$

Let E_{pq} ($0 \leq p, q \leq n+1$) be the matrix unit in $M(n+2, \mathbb{R})$. We define the following elements of the Lie algebra $\mathfrak{g} = \mathfrak{o}(n+1, 1)$:

$$\begin{aligned} (2.1) \quad X_{pq} &:= -E_{pq} + E_{qp} & (1 \leq p < q \leq n), \\ H_0 &:= E_{0,n+1} + E_{n+1,0}, \\ C_\ell^+ &:= E_{\ell,0} - E_{\ell,n+1} - E_{0,\ell} - E_{n+1,\ell} & (1 \leq \ell \leq n), \\ C_\ell^- &:= E_{\ell,0} + E_{\ell,n+1} - E_{0,\ell} + E_{n+1,\ell} & (1 \leq \ell \leq n), \\ (2.2) \quad N_\ell^+ &:= \frac{1}{2}C_\ell^+ \text{ and } N_\ell^- := C_\ell^- & (1 \leq \ell \leq n). \end{aligned}$$

Then $\{N_\ell^+\}_{\ell=1}^n$, $\{N_\ell^-\}_{\ell=1}^n$, and $\{X_{pq}\}_{1 \leq p < q \leq n} \cup \{H_0\}$ form bases of the Lie algebras $\mathfrak{n}_+(\mathbb{R}) := \text{Ker}(\text{ad}(H_0) - \text{id})$, $\mathfrak{n}_-(\mathbb{R}) := \text{Ker}(\text{ad}(H_0) + \text{id})$, and $\mathfrak{m}(\mathbb{R}) + \mathfrak{a}(\mathbb{R}) = \mathfrak{o}(n) + \mathfrak{o}(1, 1) = \text{Ker}(\text{ad}(H_0))$, respectively. We note that the normalization of N_ℓ^+ and N_ℓ^- in (2.2) is not symmetric. A simple computation shows

$$(2.3) \quad [N_k^+, N_\ell^-] = X_{k\ell} - \delta_{k\ell} H_0.$$

We define the isotropic cone (*light cone*) by

$$\Xi := \{x \in \mathbb{R}^{n+2} \setminus \{0\} : Q_{n+1,1}(x) = 0\},$$

which is clearly invariant under the dilation of the multiplicative group $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$. Then the projection

$$\Xi \longrightarrow S^n, \quad x \mapsto \frac{1}{x_{n+1}} {}^t(x_0, \dots, x_n)$$

induces a bijection $\Xi/\mathbb{R}^\times \xrightarrow{\sim} S^n$. The group G acts linearly on the isotropic cone Ξ , and conformally on $\Xi/\mathbb{R}^\times \simeq S^n$, endowed with the standard Riemannian metric. We set

$$\xi^\pm := {}^t(\pm 1, 0, \dots, 0, 1) \in \Xi.$$

Let P be the isotropy subgroup of $[\xi^+] \in \Xi/\mathbb{R}^\times$. Then P is a parabolic subgroup with Levi decomposition $P = MAN_+$ of the disconnected group $G = O(n+1, 1)$, where $A := \exp(\mathbb{R}H_0)$, $N_+ := \exp(\mathfrak{n}_+(\mathbb{R}))$ and

$$M := \left\{ \begin{pmatrix} b & & \\ & B & \\ & & b \end{pmatrix} : B \in O(n), b \in O(1) \right\} \simeq O(n) \times O(1).$$

For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, we set

$$Q_n(x) \equiv Q_{n,0}(x) := \sum_{\ell=1}^n x_\ell^2.$$

Let $N_- := \exp(\mathfrak{n}_-(\mathbb{R}))$. We define a diffeomorphism $n_-: \mathbb{R}^n \xrightarrow{\sim} N_-$ by

$$n_-(x) := \exp \left(\sum_{\ell=1}^n x_\ell N_\ell^- \right) = I_{n+2} + \begin{pmatrix} -\frac{1}{2}Q_n(x) & -{}^tx & -\frac{1}{2}Q_n(x) \\ x & 0 & x \\ \frac{1}{2}Q_n(x) & {}^tx & \frac{1}{2}Q_n(x) \end{pmatrix},$$

which gives the coordinates on the open Bruhat cell $N_- \cdot o \subset G/P \simeq \Xi/\mathbb{R}^\times \simeq S^n$:

$$(2.4) \quad \iota: \mathbb{R}^n \longrightarrow S^n, \quad x = {}^t(x_1, \dots, x_n) \mapsto \frac{1}{1 + Q_n(x)} {}^t(1 - Q_n(x), 2x_1, \dots, 2x_n),$$

because $n_-(x)\xi^+ = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -Q_n(x) \\ 2x \\ Q_n(x) \end{pmatrix}$. We note that the immersion ι is nothing but the inverse of the stereographic projection:

$$(2.5) \quad p: S^n \setminus \{[\xi^-]\} \longrightarrow \mathbb{R}^n, \quad \omega = {}^t(\omega_0, \dots, \omega_n) \mapsto \frac{1}{1 + \omega_0} {}^t(\omega_1, \dots, \omega_n),$$

where we recall $[\xi^-] = {}^t(-1, 0, \dots, 0) \in \Xi/\mathbb{R}^\times \simeq S^n$. For $\lambda \in \mathbb{C}$, we define a one-dimensional representation \mathbb{C}_λ of A normalized by

$$(2.6) \quad A \longrightarrow \mathbb{C}^\times, \quad a = e^{tH_0} \mapsto a^\lambda := e^{\lambda t}.$$

Given an irreducible finite-dimensional representation (σ, V) of $M \simeq O(n) \times O(1)$ and $\lambda \in \mathbb{C}$, we extend the outer tensor product representation $\sigma_\lambda := \sigma \boxtimes \mathbb{C}_\lambda$ of the direct product group MA to the parabolic subgroup $P = MAN_+$ by letting N_+ act trivially. Then we form an (unnormalized) principal series representation $\text{Ind}_P^G(\sigma_\lambda) \equiv \text{Ind}_P^G(\sigma \boxtimes \mathbb{C}_\lambda)$ of G on the space $(C^\infty(G) \otimes V)^P \simeq C^\infty(G, V)^P$ given by

$$\{f \in C^\infty(G, V) : f(gman) = \sigma(m)^{-1} a^{-\lambda} f(g) \text{ for all } m \in M, a \in A, g \in G\}.$$

Its flat picture (N -picture) is defined on $C^\infty(\mathbb{R}^n) \otimes V$ via the restriction to the open Bruhat cell:

$$(2.7) \quad C^\infty(G/P, \mathcal{V}) \simeq (C^\infty(G) \otimes V)^P \rightarrow C^\infty(\mathbb{R}^n) \otimes V, \quad f \mapsto (x \mapsto F(x) := f(n_-(x))).$$

We denote by $\bigwedge^i(\mathbb{C}^n)$ the representation of $O(n)$ on the i -th exterior power of the standard representation. Then, $\bigwedge^i(\mathbb{C}^n)$ ($0 \leq i \leq n$) are pairwise inequivalent, irreducible representations of $O(n)$, and $\bigwedge^n(\mathbb{C}^n)$ is isomorphic to the one-dimensional representation $\det: O(n) \rightarrow \mathbb{C}^\times$, $B \mapsto \det B$.

For $\alpha \in \mathbb{Z}/2\mathbb{Z}$ and $\lambda \in \mathbb{C}$, we denote by $\sigma_{\lambda, \alpha}^{(i)}$ the outer tensor product representation $\bigwedge^i(\mathbb{C}^n) \boxtimes (-1)^\alpha \boxtimes \mathbb{C}_\lambda$ of the Levi subgroup $L = MA \simeq O(n) \times O(1) \times \mathbb{R}$ given by

$$(B, b, a) \mapsto b^\alpha a^\lambda \bigwedge^i B \in \text{GL}_\mathbb{C}(\bigwedge^i(\mathbb{C}^n)) \text{ for } B \in O(n), \quad b \in \{\pm 1\} \simeq O(1), \quad a \in A.$$

We extend $\sigma_{\lambda, \alpha}^{(i)}$ to P by letting N_+ act trivially. We denote by $I(i, \lambda)_\alpha$ the principal series representation $\text{Ind}_P^G(\sigma_{\lambda, \alpha}^{(i)})$ of G . By a little abuse of notation, we shall also write $I(i, \lambda)_k$ for $k \in \mathbb{Z}$ instead of $I(i, \lambda)_{k \bmod 2}$.

As the composition of (2.7) with the natural identification

$$\eta: C^\infty(\mathbb{R}^n) \otimes \bigwedge^i(\mathbb{C}^n) \xrightarrow{\sim} \mathcal{E}^i(\mathbb{R}^n) \quad \text{for } 0 \leq i \leq n,$$

the flat picture of the principal series representation $I(i, \lambda)_\alpha$ is realized in $\mathcal{E}^i(\mathbb{R}^n)$:

$$(2.8) \quad \iota_\lambda^{(i)}: I(i, \lambda)_\alpha \hookrightarrow \mathcal{E}^i(\mathbb{R}^n), \quad f \mapsto F,$$

where $F(x) = \eta(f(n_-(x)))$.

Remark 2.1. The central element $-I_{n+2}$ of G acts on $I(i, \lambda)_\alpha$ as scalar multiplication by $(-1)^{i+\alpha}$. We shall see in Remark 2.4 that $I(i, \lambda)_\alpha$ appears as a representation of the conformal group $\text{Conf}(S^n)$ only when $i + \alpha \equiv 0 \bmod 2$.

We note that $G = O(n+1, 1)$ has four connected components. Let G_0 denote the identity component of G . Then we have $G/G_0 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Accordingly, there are four one-dimensional representations of G ,

$$(2.9) \quad \chi_{ab}: G \longrightarrow \{\pm 1\}$$

for $a, b \in \{\pm 1\} \equiv \{\pm 1\}$ such that

$$\chi_{ab}(\text{diag}(-1, 1, \dots, 1)) = a, \quad \chi_{ab}(\text{diag}(1, \dots, 1, -1)) = b.$$

We note that $\chi_{--} = \det$. Then the restriction of χ_{--} to $M \simeq O(n) \times O(1)$ is given by the outer tensor product:

$$(2.10) \quad \chi_{--}|_M \simeq \det \boxtimes \mathbb{1}.$$

In view of the isomorphism of $O(n)$ -modules:

$$(2.11) \quad \bigwedge^i(\mathbb{C}^n) \otimes \det \simeq \bigwedge^{n-i}(\mathbb{C}^n),$$

we get a P -isomorphism (with trivial N_+ -action):

$$\sigma_{\lambda, \alpha}^{(i)} \otimes \chi_{--}|_P \simeq \sigma_{\lambda, \alpha}^{(n-i)}.$$

Therefore, we have a natural isomorphism as G -modules:

$$\begin{aligned} I(i, \lambda)_\alpha \otimes \chi_{--} &\simeq \text{Ind}_P^G \left(\sigma_{\lambda, \alpha}^{(i)} \otimes \chi_{--}|_P \right) \\ &\simeq \text{Ind}_P^G \left(\sigma_{\lambda, \alpha}^{(n-i)} \right) \\ &\simeq I(n-i, \lambda)_\alpha. \end{aligned}$$

Thus we have proved:

Lemma 2.2. *Let $0 \leq i \leq n$, $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{Z}/2\mathbb{Z}$. Then there is a natural G -isomorphism:*

$$I(i, \lambda)_\alpha \otimes \chi_{--} \simeq I(n-i, \lambda)_\alpha.$$

2.2. Conformal view on principal series representations of $O(n+1, 1)$. Since the group $G = O(n+1, 1)$ is a double cover of the conformal group of S^n ($n \geq 2$), and since $S^n \simeq G/P$, we may compare the two families of representations of $G = O(n+1, 1)$: the family of conformal representations $\varpi_{u, \delta}^{(i)}$ and the principal series representations $I(i, \lambda)_\alpha$. The correspondence is classically known for the connected component G_0 of G (see [18] for instance). For disconnected groups G , we have the following:

Proposition 2.3. *Let $G = O(n+1, 1)$ with $n \geq 2$ and $0 \leq i \leq n$, $u \in \mathbb{C}$. For $\delta \in \mathbb{Z}/2\mathbb{Z}$, we have the following isomorphism of G -modules:*

$$\varpi_{u, \delta}^{(i)} \simeq \begin{cases} I(i, u+i)_i & \text{if } \delta = 0; \\ I(n-i, u+i)_{n-i} & \text{if } \delta = 1. \end{cases}$$

Equivalently, for $\lambda \in \mathbb{C}$, we have the following G -isomorphisms:

$$(2.12) \quad I(i, \lambda)_i \simeq \varpi_{\lambda-i, 0}^{(i)} \simeq \varpi_{\lambda-n+i, 1}^{(n-i)}.$$

Remark 2.4. Proposition 2.3 implies that principal series representations $I(\ell, \lambda)_\alpha$ with $\alpha \equiv \ell \pmod{2}$ are sufficient for the description of conformal representations $\varpi_{u,\delta}^{(i)}$ on differential forms on S^n .

Proof of Proposition 2.3. We shall show a G -isomorphism:

$$(2.13) \quad (\varpi_{u,\delta}^{(i)}, \mathcal{E}^i(S^n)) \simeq \text{Ind}_P^G ((\wedge^i(\mathbb{C}^n) \otimes \det^\delta) \boxtimes (-1)^{i+n\delta} \boxtimes \mathbb{C}_{u+i}).$$

Since the cotangent bundle of $X = G/P$ can be seen as a G -homogeneous bundle $G \times_P \mathfrak{n}_+(\mathbb{R})$, we have an isomorphism of G -modules

$$\mathcal{E}^i(S^n) \simeq C^\infty(X, G \times_P \wedge^i \mathfrak{n}_+).$$

In our setting, $\text{ad}(H_0)$ acts on \mathfrak{n}_+ as the scalar multiplication by one, and therefore, the P -action on the exterior power $\wedge^i \mathfrak{n}_+$ is given by the outer tensor product $\wedge^i(\mathbb{C}^n) \boxtimes (-1)^i \boxtimes \mathbb{C}_i$ of $MA \simeq O(n) \times O(1) \times \mathbb{R}$ with trivial N_+ -action. Thus we get the isomorphism (2.13) in the case where $u = 0$ and $\delta = 0$. On the other hand, the orientation bundle $\sigma(X)$ is associated to the one-dimensional representation of $P = LN_+ \equiv MAN_+$ given by

$$P \rightarrow P/N_+ \simeq MA \longrightarrow \{\pm 1\}, \quad (B, b, e^{tH_0}) \mapsto b^n \det B,$$

we also get (2.13) in the $u = 0$ and $\delta = 1$ case. Finally, observe that the parameter u in the definition of the conformal representation $\varpi_{u,\delta}^{(i)}$ in (1.1) is normalized in a way that the action on volume densities corresponds to the case $u = \dim X$ (with $i = 0$ and $\delta = 0$). In our setting where $X = G/P \simeq S^n$, this coincides with $n = \text{Trace}(\text{ad}(H_0): \mathfrak{n}_+(\mathbb{R}) \rightarrow \mathfrak{n}_+(\mathbb{R})) = 2\rho$ via the normalization (2.6) that we have adopted for the principal series representations. Hence, (2.13) is verified for all $u \in \mathbb{C}$ by interpolation. By (2.11), Proposition 2.3 follows. \square

2.3. Representation theoretic properties of $(\varpi_{u,\delta}^{(i)}, \mathcal{E}^i(S^n))$. Via the isomorphism in Proposition 2.3, we can apply the general theory of representations of real reductive groups to our representations $(\varpi_{u,\delta}^{(i)}, \mathcal{E}^i(S^n))$ of the conformal group. Although the large majority of the literature in the representation theory of real reductive groups G is limited to reductive groups of the Harish-Chandra class, our group $G = O(n+1, 1)$ is disconnected and the adjoint group $\text{Ad}(G)$ is not contained in the group $\text{Int}(\mathfrak{g})$ of inner automorphisms of the complexified Lie algebra $\mathfrak{g} = \mathfrak{o}(n+2, \mathbb{C})$ if n is even. This does not cause any serious difficulties in the argument below, but we shall be careful in preparing notation for the disconnected group G .

Let $Z_G(\mathfrak{g})$ be the ring of $\text{Ad}(G)$ -invariant elements in the enveloping algebra $U(\mathfrak{g})$ of the complexified Lie algebra $\mathfrak{g} \simeq \mathfrak{o}(n+2, \mathbb{C})$. We note that $Z_G(\mathfrak{g})$ is a subalgebra of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$; it coincides with $Z(\mathfrak{g})$ if n is odd, and is of index two in $Z(\mathfrak{g})$ if n is even.

By taking the standard basis of a Cartan subalgebra \mathfrak{j} of $\mathfrak{g} = \mathfrak{o}(n+2, \mathbb{C})$, we identify \mathfrak{j} with $\mathbb{C}^{\lfloor \frac{n}{2} \rfloor + 1}$. The finite reflection group $W = \mathfrak{S}_{\lfloor \frac{n}{2} \rfloor + 1} \ltimes (\mathbb{Z}/2\mathbb{Z})^{\lfloor \frac{n}{2} \rfloor + 1}$ acts naturally on \mathfrak{j} and $\mathfrak{j}^\vee \simeq \mathbb{C}^{\lfloor \frac{n}{2} \rfloor + 1}$. We note that W coincides with the Weyl group of the root system of type $B_{\lfloor \frac{n}{2} \rfloor + 1}$ if n is odd, and contains that of type $D_{\lfloor \frac{n}{2} \rfloor + 1}$ as a subgroup of index two if n is even. Then the Harish-Chandra isomorphism for the disconnected group $G = O(n+1, 1)$ asserts a \mathbb{C} -algebra isomorphism between $Z_G(\mathfrak{g})$ and the ring $S\left(\mathbb{C}^{\lfloor \frac{n}{2} \rfloor + 1}\right)^W$ of W -invariants of the symmetric algebra $S(\mathfrak{j})$. In turn, we have a bijection (*Harish-Chandra's parametrization of infinitesimal characters*)

$$(2.14) \quad \text{Hom}_{\mathbb{C}\text{-algebra}}(Z_G(\mathfrak{g}), \mathbb{C}) \simeq \mathbb{C}^{\lfloor \frac{n}{2} \rfloor + 1} / W.$$

We normalize the Harish-Chandra isomorphism in a way that the $Z_G(\mathfrak{g})$ -infinitesimal character of the trivial one-dimensional representation $\mathbb{1}$ of G is given by

$$(2.15) \quad \rho_G := \left(\frac{n}{2}, \frac{n}{2} - 1, \dots, \frac{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor \right) \in \mathbb{C}^{\lfloor \frac{n}{2} \rfloor + 1} / W.$$

Proposition 2.5. *The $Z_G(\mathfrak{g})$ -infinitesimal character of the representation $\varpi_{u,\delta}^{(i)}$ of G on the space $\mathcal{E}^i(S^n)$ of i -forms is given by*

$$\begin{aligned} & \left(u + i - \frac{n}{2}, \underbrace{\frac{n}{2}, \frac{n}{2} - 1, \dots, \frac{n}{2} - i + 1}_i, \widehat{\frac{n}{2} - i}, \underbrace{\frac{n}{2} - i - 1, \dots, \frac{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor}_{\lfloor \frac{n}{2} \rfloor - i} \right) \quad \text{if } 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor, \\ & \left(u + i - \frac{n}{2}, \underbrace{\frac{n}{2}, \frac{n}{2} - 1, \dots, -\frac{n}{2} + i + 1}_{n-i}, \widehat{-\frac{n}{2} + i}, \underbrace{-\frac{n}{2} + i - 1, \dots, \frac{n}{2} - \left\lfloor \frac{n}{2} \right\rfloor}_{i - \lfloor \frac{n+1}{2} \rfloor} \right) \quad \text{if } \left\lfloor \frac{n+1}{2} \right\rfloor \leq i \leq n, \end{aligned}$$

in the Harish-Chandra parametrization.

In particular, $\varpi_{u,\delta}^{(i)}$ has the same infinitesimal character ρ_G with the trivial representation if $u = 0$ for all $0 \leq i \leq n$ and $\delta \in \mathbb{Z}/2\mathbb{Z}$.

By the Frobenius reciprocity, every principal series representation $I(i, \lambda)_\alpha$ contains

$$(2.16) \quad \mu^b \equiv \mu^b(i) := \bigwedge^i(\mathbb{C}^{n+1}) \boxtimes (-1)^\alpha \quad \text{and} \quad \mu^\# \equiv \mu^\#(i) := \bigwedge^{i+1}(\mathbb{C}^{n+1}) \boxtimes (-1)^{\alpha+1}$$

as K -types. We are particularly interested in the $\lambda = i$ case, for which $I(i, \lambda)_\alpha$ is reducible (except for $n = 2i$) and has $Z_G(\mathfrak{g})$ -infinitesimal character ρ_G .

We denote by $\bar{I}(i)_\alpha^b$ and $\bar{I}(i)_\alpha^\#$ the (unique) irreducible subquotients of $I(i, i)_\alpha$ containing the K -types μ^b and $\mu^\#$, respectively. Then we have G -isomorphisms

$$(2.17) \quad \bar{I}(i)_\alpha^\# \simeq \bar{I}(i+1)_{\alpha+1}^b \quad \text{for } 0 \leq i \leq n \text{ and } \alpha \in \mathbb{Z}/2\mathbb{Z}.$$

For n even, the unitary axis of $I\left(\frac{n}{2}, \lambda\right)_\alpha$ is given by $\lambda = \frac{n}{2} + \sqrt{-1}\mathbb{R}$, and $I\left(\frac{n}{2}, \frac{n}{2}\right)_\alpha$ is irreducible for both $\alpha \equiv 0$ and 1 in $\mathbb{Z}/2\mathbb{Z}$. In particular, we have

$$(2.18) \quad \bar{I}\left(\frac{n}{2}\right)_\alpha^b = \bar{I}\left(\frac{n}{2}\right)_\alpha^\# \quad \text{for } \alpha \in \mathbb{Z}/2\mathbb{Z}.$$

For $0 \leq \ell \leq n+1$ and $\delta \in \mathbb{Z}/2\mathbb{Z}$, we set

$$\Pi_{\ell,\delta} := \begin{cases} \bar{I}(\ell)_{\ell+\delta}^b & (0 \leq \ell \leq n), \\ \bar{I}(\ell-1)_{\ell+\delta+1}^\# & (1 \leq \ell \leq n+1). \end{cases}$$

In view of (2.17) and (2.18), $\Pi_{\ell,\delta}$ is well-defined and

$$(2.19) \quad \Pi_{\frac{n}{2},\delta} \simeq \Pi_{\frac{n}{2}+1,\delta},$$

when n is even.

Theorem 2.6. *Let $G = O(n+1, 1)$ ($n \geq 1$).*

- 1) *Irreducible representations of G with $Z_G(\mathfrak{g})$ -infinitesimal character ρ_G are classified as*

$$\{\Pi_{\ell,\delta} : 0 \leq \ell \leq n+1, \delta \in \mathbb{Z}/2\mathbb{Z}\}$$

with the equivalence relation (2.19) when n is even.

- 2) *There are four one-dimensional representations of G , and they are given by*

$$\{\Pi_{0,\delta}, \Pi_{n+1,\delta} : \delta \in \mathbb{Z}/2\mathbb{Z}\} (= \{\chi_{ab} : a, b \in \mathbb{Z}/2\mathbb{Z}\}).$$

- 3) *For n odd, $\Pi_{\frac{n+1}{2},\delta}$ ($\delta \in \mathbb{Z}/2\mathbb{Z}$) are discrete series representations of G . For n even, $\Pi_{\frac{n}{2},\delta} (\simeq \Pi_{\frac{n}{2}+1,\delta})$ ($\delta \in \mathbb{Z}/2\mathbb{Z}$) are tempered representations of G .*
- 4) *Every $\Pi_{\ell,\delta}$ ($0 \leq \ell \leq n+1, \delta \in \mathbb{Z}/2\mathbb{Z}$) is unitarizable.*
- 5) *Irreducible and unitarizable (\mathfrak{g}, K) -modules with nonzero (\mathfrak{g}, K) -cohomologies are exactly given as the set of the underlying (\mathfrak{g}, K) -modules of $\Pi_{\ell,\delta}$ ($0 \leq \ell \leq n+1, \delta \in \mathbb{Z}/2\mathbb{Z}$) up to the equivalence (2.19) when n is even.*
- 6) *For $0 \leq i \leq n$ with $n \neq 2i$, we have a nonsplitting exact sequence of G -modules*

$$0 \longrightarrow \Pi_{i,0} \longrightarrow \varpi_{0,0}^{(i)} \longrightarrow \Pi_{i+1,0} \longrightarrow 0.$$

For $n = 2i$, we have a G -isomorphism:

$$\varpi_{0,0}^{(i)} \simeq \Pi_{\frac{n}{2},0}.$$

Furthermore, the de Rham complex

$$\mathcal{E}^0(S^n) \xrightarrow{d} \mathcal{E}^1(S^n) \xrightarrow{d} \mathcal{E}^2(S^n) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{E}^n(S^n) \xrightarrow{d} \{0\}$$

yields a family of intertwining operators for $(\varpi_{0,0}^{(i)}, \mathcal{E}^i(S^n))$, and

$$\begin{aligned} \text{Ker}(d: \mathcal{E}^i(S^n) \longrightarrow \mathcal{E}^{i+1}(S^n)) &= \begin{cases} \Pi_{i,0} & (0 \leq i \leq n-1), \\ \mathcal{E}^i(S^n) & (i = n), \end{cases} \\ \text{Image}(d: \mathcal{E}^{i-1}(S^n) \longrightarrow \mathcal{E}^i(S^n)) &= \begin{cases} \{0\} & (i = 0), \\ \Pi_{i,0} & (1 \leq i \leq n), \end{cases} \end{aligned}$$

giving rise to

$$H_{\text{de Rham}}^i(S^n; \mathbb{C}) \simeq \begin{cases} \Pi_{0,0} & (i = 0), \\ \{0\} & (1 \leq i \leq n-1), \\ \Pi_{n,0} & (i = n). \end{cases}$$

as G -modules.

2.4. Differential symmetry breaking operators for principal series.

This section gives a group theoretic reformulation of the main results stated in Introduction (see Theorem 1.1 and Theorems 1.5-1.8) via the isomorphism in Proposition 2.3.

Let us realize $G' = O(n, 1)$ in G as the stabilizer of the point ${}^t(0, \dots, 0, 1, 0) \in \mathbb{R}^{n+2}$. Then G' leaves $\Xi \cap \{x_n = 0\}$ invariant, and acts conformally on the totally geodesic hypersphere $S^{n-1} = \{(y_0, \dots, y_n) \in S^n : y_n = 0\} \simeq (\Xi \cap \{x_n = 0\})/\mathbb{R}^\times$. The isotropy subgroup of $[\xi^+] \in S^{n-1}$ is a parabolic subgroup $P' = P \cap G'$, which has a Langlands decomposition $P' = M'AN'_+$ with $M' = M \cap G' \simeq O(n-1) \times O(1)$ and A being the same split abelian subgroup as in P . The Lie algebra $\mathfrak{n}'_+(\mathbb{R})$ of N'_+ is given by

$$\mathfrak{n}'_+(\mathbb{R}) = \sum_{k=1}^{n-1} \mathbb{R}N_k^+.$$

Given a representation (σ, V) of $M \simeq O(n) \times O(1)$ and $\lambda \in \mathbb{C}$, we defined in Section 2.1 the principal series representation $\text{Ind}_P^G(\sigma_\lambda) \equiv \text{Ind}_P^G(\sigma \boxtimes \mathbb{C}_\lambda)$ of $G = O(n+1, 1)$. Similarly, for a given representation (τ, W) of $M' \simeq O(n-1) \times O(1)$ and $\nu \in \mathbb{C}$, we define the principal series representation $\text{Ind}_{P'}^{G'}(\tau_\nu) \equiv \text{Ind}_{P'}^{G'}(\tau \boxtimes \mathbb{C}_\nu)$ of $G' = O(n, 1)$, and consider its N -picture on $C^\infty(\mathbb{R}^{n-1}) \otimes W$. Then differential symmetry breaking operators from $\text{Ind}_P^G(\sigma_\lambda)$ to $\text{Ind}_{P'}^{G'}(\tau_\nu)$ are given as differential operators $C^\infty(\mathbb{R}^n) \otimes V \rightarrow C^\infty(\mathbb{R}^{n-1}) \otimes W$, namely, $\text{Hom}_{\mathbb{C}}(V, W)$ -valued differential operators from \mathbb{R}^n to \mathbb{R}^{n-1} in the N -picture.

As in the case of $G = O(n+1, 1)$, $\tau_{\nu, \beta}^{(j)}$ ($0 \leq j \leq n-1$, $\nu \in \mathbb{C}$, $\beta \in \mathbb{Z}/2\mathbb{Z}$) denotes the representation of $P' = M'AN'_+$ such that $M'A \simeq O(n-1) \times O(1) \times A$ acts as the outer tensor product representation on $\bigwedge^j(\mathbb{C}^{n-1}) \boxtimes (-1)^\beta \boxtimes \mathbb{C}_\nu$ and N'_+ acts trivially. Then we define the principal series representation of $G' = O(n, 1)$ by $J(j, \nu)_\beta := \text{Ind}_{P'}^{G'}(\tau_{\nu, \beta}^{(j)})$. First we prove a duality theorem for symmetry breaking operators:

Theorem 2.7 (duality theorem). *Let $0 \leq i \leq n$, $0 \leq j \leq n-1$, $\lambda, \nu \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$. Then*

$$(2.20) \quad \text{Diff}_{G'}(I(i, \lambda)_\alpha, J(j, \nu)_\beta) \simeq \text{Diff}_{G'}(I(n-i, \lambda)_\alpha, J(n-1-j, \nu)_\beta).$$

Proof. Applying Lemma 2.2 to G and G' , we have natural G - and G' -isomorphisms:

$$\begin{aligned} I(i, \lambda)_\alpha \otimes \chi_{--} &\simeq I(n-i, \lambda)_\alpha, \\ J(j, \nu)_\beta \otimes \chi_{--}|_{G'} &\simeq J(n-1-j, \nu)_\beta. \end{aligned}$$

Therefore we have the following natural bijections:

$$\begin{aligned} \text{Hom}_{G'}(I(i, \lambda)_\alpha, J(j, \nu)_\beta) &\simeq \text{Hom}_{G'}(I(i, \lambda)_\alpha \otimes \chi_{--}, J(j, \nu)_\beta \otimes \chi_{--}|_{G'}) \\ &\simeq \text{Hom}_{G'}(I(n-i, \lambda)_\alpha, J(n-1-j, \nu)_\beta). \end{aligned}$$

The above isomorphisms preserve differential operators for the geometric realizations of principal series on the Fréchet spaces of smooth sections of equivariant vector bundles over real flag varieties. Thus we have shown the isomorphism (2.20). \square

In order to avoid possible confusion with the parameter for the conformal representation $(\varpi_{u,\delta}^{(i)}, \mathcal{E}^i(S^n))$, it is convenient to introduce another notation for the differential symmetry breaking operators between principal series representations in the N -picture. The notation below follows from [22] which treats both local (*i.e.*, differential) and nonlocal symmetry breaking operators.

For $\lambda, \nu \in \mathbb{C}$ with $\nu - \lambda \in \mathbb{N}$, we define (scalar-valued) differential operators $\tilde{\mathbb{C}}_{\lambda,\nu}: C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^{n-1})$ by

$$\begin{aligned} (2.21) \quad \tilde{\mathbb{C}}_{\lambda,\nu} &:= \text{Rest}_{x_n=0} \circ \left(I_{\nu-\lambda} \tilde{C}_{\nu-\lambda}^{\lambda-\frac{n-1}{2}} \right) \left(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n} \right) \\ &= \text{Rest}_{x_n=0} \circ \mathcal{D}_{\nu-\lambda}^{\lambda-\frac{n-1}{2}}, \end{aligned}$$

where $(I_\ell \tilde{C}_\ell^\mu)(x, y) = x^{\frac{\ell}{2}} \tilde{C}_\ell^\mu\left(\frac{y}{\sqrt{x}}\right)$ is a polynomial of two variables associated with the renormalized Gegenbauer polynomial (see (14.3)) and the corresponding differential operator \mathcal{D}_ℓ^μ is given by (1.2).

For example, we have

$$\begin{aligned} \tilde{\mathbb{C}}_{\lambda+1,\nu-1} &= \text{Rest}_{x_n=0} \circ \mathcal{D}_{\nu-\lambda-2}^{\lambda-\frac{n-3}{2}}, \\ \tilde{\mathbb{C}}_{\lambda+1,\nu} &= \text{Rest}_{x_n=0} \circ \mathcal{D}_{\nu-\lambda-1}^{\lambda-\frac{n-3}{2}}, \\ \tilde{\mathbb{C}}_{\lambda,\nu-1} &= \text{Rest}_{x_n=0} \circ \mathcal{D}_{\nu-\lambda-1}^{\lambda-\frac{n-1}{2}}. \end{aligned}$$

Next, for $\lambda, \nu \in \mathbb{C}$ with $\nu - \lambda \in \mathbb{N}$, we define (matrix-valued) differential operators

$$\mathbb{C}_{\lambda,\nu}^{i,j}: \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^j(\mathbb{R}^{n-1})$$

as follows: they are essentially the same with the operators $\mathcal{D}_{u,a}^{i \rightarrow j}$ or $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow j}$, respectively, introduced in Chapter 1. To be precise, in all the cases below, the parameters for $\mathbb{C}_{\lambda,\nu}^{i,j}$ or $\tilde{\mathbb{C}}_{\lambda,\nu}^{i,j}$ are taken as

$$(2.22) \quad \mathbb{C}_{\lambda,\nu}^{i,j} = \mathcal{D}_{u,a}^{i \rightarrow j}, \quad \tilde{\mathbb{C}}_{\lambda,\nu}^{i,j} = \tilde{\mathcal{D}}_{u,a}^{i \rightarrow j}$$

with $a = \nu - \lambda$ and $u = \lambda - i$. The parameters of the operators $\mathbb{C}_{\lambda,\nu}^{i,j}$ or $\tilde{\mathbb{C}}_{\lambda,\nu}^{i,j}$ indicate that these operators induce symmetry breaking operators from $I(i, \lambda)_\alpha$ to $I(i, \lambda)_\beta$ when $\nu - \lambda \equiv \beta - \alpha \pmod{2}$, whereas the parameters of $\mathcal{D}_{u,a}^{i \rightarrow j}$ or $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow j}$ indicate that $a \in \mathbb{N}$ is the order of the differential operators and $u \in \mathbb{C}$ is normalized in a way that $u = 0$ gives the untwisted case.

For $j = i$ with $0 \leq i \leq n - 1$, we recall from (1.6) and (1.7) the formulæ of $\mathcal{D}_{u,a}^{i \rightarrow i}$, and set

$$(2.23) \quad \begin{aligned} \mathbb{C}_{\lambda,\nu}^{i,i} &:= \mathcal{D}_{\lambda-i,\nu-\lambda}^{i \rightarrow i} \\ &= \tilde{\mathbb{C}}_{\lambda+1,\nu-1} d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* - \gamma(\lambda - \frac{n}{2}, \nu - \lambda) \tilde{\mathbb{C}}_{\lambda,\nu-1} d_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}} + \frac{1}{2}(\nu - i) \tilde{\mathbb{C}}_{\lambda,\nu}, \end{aligned}$$

$$(2.24) \quad = -d_{\mathbb{R}^{n-1}}^* d_{\mathbb{R}^{n-1}} \tilde{\mathbb{C}}_{\lambda+1,\nu-1} + \gamma(\lambda - \frac{n-1}{2}, \nu - \lambda) \tilde{\mathbb{C}}_{\lambda+1,\nu} \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n} + \frac{\lambda - i}{2} \tilde{\mathbb{C}}_{\lambda,\nu},$$

in the flat coordinates. The equalities

$$(2.23) = (2.24) = (-1)^{n-1} *_{\mathbb{R}^{n-1}} \circ \mathbb{C}_{\lambda,\nu}^{n-i,n-i-1} \circ (*_{\mathbb{R}^n})^{-1}$$

will be proved in Proposition 10.3.

For $j = i - 1$ with $1 \leq i \leq n$, we recall from (1.4) and (1.5) the formulæ of $\mathcal{D}_{u,a}^{i \rightarrow i-1}$, and set

$$(2.25) \quad \begin{aligned} \mathbb{C}_{\lambda,\nu}^{i,i-1} &:= \mathcal{D}_{\lambda-i,\nu-\lambda}^{i \rightarrow i-1} \\ &= -\tilde{\mathbb{C}}_{\lambda+1,\nu-1} d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} - \gamma(\lambda - \frac{n-1}{2}, \nu - \lambda) \tilde{\mathbb{C}}_{\lambda+1,\nu} d_{\mathbb{R}^n}^* + \frac{1}{2}(\lambda + i - n) \tilde{\mathbb{C}}_{\lambda,\nu} \iota_{\frac{\partial}{\partial x_n}} \\ (2.26) \quad &= -\tilde{\mathbb{C}}_{\lambda+1,\nu-1} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n} + \frac{1}{2}(\nu - n + i) \tilde{\mathbb{C}}_{\lambda,\nu} \iota_{\frac{\partial}{\partial x_n}} - \gamma(\lambda - \frac{n}{2}, \nu - \lambda) d_{\mathbb{R}^{n-1}}^* \tilde{\mathbb{C}}_{\lambda,\nu-1}. \end{aligned}$$

Then Proposition 1.4 means that

$$(2.27) \quad \mathbb{C}_{\lambda,\nu}^{i,i} = 0 \text{ if and only if } \lambda = \nu = i \text{ or } \nu = i = 0,$$

$$(2.28) \quad \mathbb{C}_{\lambda,\nu}^{i,i-1} = 0 \text{ if and only if } \lambda = \nu = n - i \text{ or } \nu = n - i = 0.$$

We note that $\mathbb{C}_{\lambda,\nu}^{0,0} = \frac{1}{2}\nu\tilde{\mathbb{C}}_{\lambda,\nu}$. As in (1.9) and (1.10), we renormalize these operators by

(2.29)

$$\tilde{\mathbb{C}}_{\lambda,\nu}^{i,i} := \begin{cases} \text{Rest}_{x_n=0} & \text{if } \lambda = \nu, \\ \tilde{\mathbb{C}}_{\lambda,\nu} & \text{if } i = 0, \\ \mathbb{C}_{\lambda,\nu}^{i,i} & \text{otherwise,} \end{cases} \quad \text{and} \quad \tilde{\mathbb{C}}_{\lambda,\nu}^{i,i-1} := \begin{cases} \text{Rest}_{x_n=0} \circ \iota_{\frac{\partial}{\partial x_n}} & \text{if } \lambda = \nu, \\ \tilde{\mathbb{C}}_{\lambda,\nu} \circ \iota_{\frac{\partial}{\partial x_n}} & \text{if } i = n, \\ \mathbb{C}_{\lambda,\nu}^{i,i-1} & \text{otherwise.} \end{cases}$$

Then $\tilde{\mathbb{C}}_{\lambda,\nu}^{i,i}$ ($0 \leq i \leq n-1$) and $\tilde{\mathbb{C}}_{\lambda,\nu}^{i,i-1}$ ($1 \leq i \leq n$) are nonzero differential operators of order $\nu - \lambda$ for any $\lambda, \nu \in \mathbb{C}$ with $\nu - \lambda \in \mathbb{N}$.

The differential symmetry breaking operator $\tilde{\mathbb{C}}_{\lambda,\nu}^{i,i+1}$ is defined by

$$(2.30) \quad \tilde{\mathbb{C}}_{\lambda,\nu}^{i,i+1} := \tilde{\mathcal{D}}_{\lambda-i,\nu-\lambda}^{i \rightarrow i+1} = \text{Rest}_{x_n=0} \circ \left(I_{i-\lambda} \tilde{C}_{i-\lambda}^{\lambda-i-\frac{n-1}{2}} \right) \left(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n} \right) d_{\mathbb{R}^n},$$

but only when $(\lambda, \nu) = (i, i+1)$ for $1 \leq i \leq n-2$ or $\lambda \in -\mathbb{N}$, $\nu = 1$ for $i = 0$. Explicitly, these operators take the following form:

$$\begin{aligned} \tilde{\mathbb{C}}_{i,i+1}^{i,i+1} &= \text{Rest}_{x_n=0} \circ d_{\mathbb{R}^n} && \text{for } 1 \leq i \leq n-2, \\ \tilde{\mathbb{C}}_{\lambda,1}^{0,1} &= \text{Rest}_{x_n=0} \circ \left(I_{-\lambda} \tilde{C}_{-\lambda}^{\lambda-\frac{n-1}{2}} \right) \left(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n} \right) d_{\mathbb{R}^n} \\ &= d_{\mathbb{R}^{n-1}} \circ \tilde{\mathbb{C}}_{\lambda,0} && \text{for } \lambda \in -\mathbb{N}. \end{aligned}$$

Similarly, we define

(2.31)

$$\tilde{\mathbb{C}}_{\lambda,\nu}^{i,i-2} := \tilde{\mathcal{D}}_{\lambda-i,\nu-\lambda}^{i \rightarrow i-2} = \text{Rest}_{x_n=0} \circ \left(I_{n-i-\lambda} \tilde{C}_{n-i-\lambda}^{\lambda-i+\frac{n+1}{2}} \right) \left(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n} \right) \circ \iota_{\frac{\partial}{\partial x_n}} \circ d_{\mathbb{R}^n},$$

but only when $(\lambda, \nu) = (n-i, n-i+1)$ ($2 \leq i \leq n-1$) or $\lambda \in -\mathbb{N}$, $\nu = 1$ ($i = n$). Explicitly, these operators take the following form:

$$\begin{aligned} \tilde{\mathbb{C}}_{n-i,n-i+1}^{i,i-2} &= \text{Rest}_{x_n=0} \circ \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n}^* && \text{for } 2 \leq i \leq n-1, \\ \tilde{\mathbb{C}}_{\lambda,1}^{n,n-2} &= \text{Rest}_{x_n=0} \circ \left(I_{-\lambda} \tilde{C}_{-\lambda}^{\lambda-\frac{n-1}{2}} \right) \left(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n} \right) \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n}^* \\ &= -d_{\mathbb{R}^{n-1}}^* \circ \tilde{\mathbb{C}}_{\lambda,0}^{n,n-1} && \text{for } \lambda \in -\mathbb{N}. \end{aligned}$$

To see the second equality, we use some elementary commutation relations which will be given in Lemma 8.14 (2) and Lemma 8.15 (2) among others.

We are ready to give a classification of symmetry breaking operators from the principal series representation $I(i, \lambda)_\alpha$ of $G = O(n+1, 1)$ to the principal series representation $J(j, \nu)_\beta$ of the subgroup $G' = O(n, 1)$.

Theorem 2.8. *Let $n \geq 3$. Suppose $0 \leq i \leq n$, $0 \leq j \leq n-1$, $\lambda, \nu \in \mathbb{C}$, and $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$. The following three conditions on 6-tuple $(i, j, \lambda, \nu, \alpha, \beta)$ are equivalent:*

- (i) $\text{Diff}_{O(n,1)}(I(i, \lambda)_\alpha, J(j, \nu)_\beta) \neq \{0\}$.
- (ii) $\dim \text{Diff}_{O(n,1)}(I(i, \lambda)_\alpha, J(j, \nu)_\beta) = 1$.
- (iii) *The 6-tuple belongs to one of the following six cases:*
 - Case 1. $j = i-2$, $2 \leq i \leq n-1$, $(\lambda, \nu) = (n-i, n-i+1)$, $\beta \equiv \alpha+1 \pmod{2}$.
 - Case 1'. $(i, j) = (n, n-2)$, $-\lambda \in \mathbb{N}$, $\nu = 1$, $\beta \equiv \alpha + \lambda + 1 \pmod{2}$.
 - Case 2. $j = i-1$, $1 \leq i \leq n$, $\nu - \lambda \in \mathbb{N}$, $\beta - \alpha \equiv \nu - \lambda \pmod{2}$.
 - Case 3. $j = i$, $0 \leq i \leq n-1$, $\nu - \lambda \in \mathbb{N}$, $\beta - \alpha \equiv \nu - \lambda \pmod{2}$.
 - Case 4. $j = i+1$, $1 \leq i \leq n-2$, $(\lambda, \nu) = (i, i+1)$, $\beta \equiv \alpha + 1 \pmod{2}$.
 - Case 4'. $(i, j) = (0, 1)$, $-\lambda \in \mathbb{N}$, $\nu = 1$, $\beta \equiv \alpha + \lambda + 1 \pmod{2}$.

Theorem 2.9. *Retain the setting and notations as in Theorem 2.8. Then the following differential operators from $\mathcal{E}^i(\mathbb{R}^n)$ to $\mathcal{E}^j(\mathbb{R}^{n-1})$ in the flat picture extend to a nonzero $O(n, 1)$ -homomorphism from $I(i, \lambda)_\alpha$ to $J(j, \nu)_\beta$:*

- Cases 1 and 1'. $\tilde{\mathbb{C}}_{n-i, n-i+1}^{i, i-2}$ ($2 \leq i \leq n-1$), $\tilde{\mathbb{C}}_{\lambda, 1}^{n, n-2}$;
- Case 2. $\tilde{\mathbb{C}}_{\lambda, \nu}^{i, i-1}$ ($1 \leq i \leq n$);
- Case 3. $\tilde{\mathbb{C}}_{\lambda, \nu}^{i, i}$ ($0 \leq i \leq n-1$);
- Cases 4 and 4'. $\tilde{\mathbb{C}}_{i, i+1}^{i, i+1}$ ($1 \leq i \leq n-2$), $\tilde{\mathbb{C}}_{\lambda, 1}^{0, 1}$.

Conversely, any differential symmetry breaking operator from $I(i, \lambda)_\alpha$ to $J(j, \nu)_\beta$ in Theorem 2.8 is proportional to one of these operators.

The proof of Theorem 2.8 is reduced to solving the F-system, which we carry out in Chapter 6 for Case 2, Chapter 7 (Theorem 7.1) for Cases 4 and 4'. The remaining cases (*i.e.* Cases 3, 1 and 1') in Theorem 2.8 follows from Cases 2, 4, and 4', respectively, by the duality theorem (Theorem 2.7). In summary, Cases 1 and 1', Case 2, Case 3, and Cases 4 and 4' in Theorem 2.8 are stated and proved in Theorem 7.2, 6.3, 6.4, and 7.1, respectively. The proof of Theorem 2.9 will be completed in Chapter 10.

In Chapter 11, we shall see that Theorem 1.1 is derived from Theorem 2.8 via the isomorphism in Proposition 2.3. Theorems 1.1, 1.5, 1.6, and 1.8 are obtained from Theorem 2.9 (see Section 11.4).

2.5. Symmetry breaking operators for connected group $SO_0(n, 1)$. So far we have dealt with the *disconnected* group $G' = O(n, 1)$ in studying symmetry breaking operators. Results for the *connected* group $G'_0 = SO_0(n, 1)$ (or equivalently, for conformal vector fields on S^n along the submanifold S^{n-1}) can be deduced from those in the disconnected case.

In this section we explain a trick for the reduction to the connected case. Let $G_0 = SO_0(n+1, 1)$ be the identity component of $G = O(n+1, 1)$, and $G'_0 = SO_0(n, 1)$ be that of $G' = O(n, 1)$.

The connected group G_0 acts transitively on G/P , and we have a natural isomorphism

$$G_0/P_0 \xrightarrow{\sim} G/P (\simeq S^n),$$

where $P_0 := P \cap G_0 = M_0 A N_+$ is a parabolic subgroup of G_0 . Then both P_0 and $M_0 \simeq SO(n)$ are connected. For $0 \leq i \leq n$ and $\lambda \in \mathbb{C}$, we write $I(i, \lambda)$ for the (unnormalized) induced representation $\text{Ind}_{P_0}^{G_0} (\bigwedge^i(\mathbb{C}^n) \boxtimes \mathbb{C}_\lambda)$ of G_0 . (We note that $\bigwedge^i(\mathbb{C}^n)$ is reducible as an M_0 -module if and only if $n = 2i$, but we do not enter this point here.) We recall from Section 2.1 that $I(i, \lambda)_\alpha$ ($\alpha \in \mathbb{Z}/2\mathbb{Z}$) is a principal series representation of G , which we may realize in the space $\mathcal{E}^i(S^n)$ of i -forms on S^n . The restriction to G_0 is independent of $\alpha \in \mathbb{Z}/2\mathbb{Z}$, and we have isomorphisms as G_0 -modules:

$$(2.32) \quad I(i, \lambda)_\alpha|_{G_0} \simeq I(i, \lambda) \simeq I(n-i, \lambda).$$

Analogous notation will be applied to the subgroup $G'_0 = SO_0(n, 1)$. In particular, $J(j, \nu)$ ($0 \leq j \leq n-1$, $\nu \in \mathbb{C}$) denotes the (unnormalized) induced representation $\text{Ind}_{P'_0}^{G'_0} (\bigwedge^j(\mathbb{C}^{n-1}) \boxtimes \mathbb{C}_\nu)$, and we have isomorphisms as G'_0 -modules:

$$(2.33) \quad J(j, \nu)_\beta|_{G'_0} \simeq J(j, \nu) \simeq J(n-1-j, \nu),$$

defined on $\mathcal{E}^j(S^{n-1})$.

In what follows, we set

$$\tilde{i} := n-i, \quad \tilde{j} := n-1-j.$$

We are ready to state the results on differential symmetry breaking operators for the *connected* subgroup $G'_0 = SO_0(n, 1)$:

Theorem 2.10. *Suppose $0 \leq i \leq n$, $0 \leq j \leq n-1$, and $\lambda, \nu \in \mathbb{C}$.*

(1) *There are natural bijections:*

$$\begin{aligned} \text{Diff}_{SO_0(n,1)}(I(i, \lambda), J(j, \nu)) &\simeq \text{Diff}_{SO_0(n,1)}(I(\tilde{i}, \lambda), J(j, \nu)) \\ &\simeq \text{Diff}_{SO_0(n,1)}(I(i, \lambda), J(\tilde{j}, \nu)) \\ &\simeq \text{Diff}_{SO_0(n,1)}(I(\tilde{i}, \lambda), J(\tilde{j}, \nu)). \end{aligned}$$

(2) *The above space is nonzero only when $\nu - \lambda \in \mathbb{N}$. Assume now $\nu - \lambda \in \mathbb{N}$.*

We fix $\alpha \in \mathbb{Z}/2\mathbb{Z}$ and set $\beta := \alpha + \nu - \lambda \bmod 2$. Then we have

$$\begin{aligned} \text{Diff}_{SO_0(n,1)}(I(i, \lambda), J(j, \nu)) &\simeq \text{Diff}_{O(n,1)}(I(i, \lambda)_\alpha, J(j, \nu)_\beta) \oplus \text{Diff}_{O(n,1)}(I(\tilde{i}, \lambda)_\alpha, J(j, \nu)_\beta) \\ &\simeq \text{Diff}_{O(n,1)}(I(i, \lambda)_\alpha, J(j, \nu)_\beta) \oplus \text{Diff}_{O(n,1)}(I(i, \lambda)_\alpha, J(\tilde{j}, \nu)_\beta). \end{aligned}$$

The second statement shows that the classification and construction of differential symmetry breaking operators for the connected group $G'_0 = SO_0(n, 1)$ are deduced from the one for the disconnected case that we have given in Theorems 2.8 and 2.9.

Proof. The first statement follows directly from (2.32) and (2.33). To see the second statement, we set

$$\begin{aligned} S &:= \{0, 1, \dots, n\} \times \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}, & I(s) &:= I(i, \lambda)_\alpha \quad \text{for } s = (i, \lambda, \alpha) \in \mathbb{Z}/2\mathbb{Z}, \\ T &:= \{0, 1, \dots, n-1\} \times \mathbb{C} \times \mathbb{Z}/2\mathbb{Z}, & J(t) &:= J(j, \nu)_\beta \quad \text{for } t = (j, \nu, \beta) \in \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

We recall from (2.9) that the quotient groups are given by

$$G'/G'_0 \simeq G/G_0 \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z},$$

and the set of their one-dimensional representations is given by

$$(G'/G'_0)^\wedge \simeq (G/G_0)^\wedge = \{\chi_{ab} : a, b \in \{\pm\}\}.$$

By abuse of notation, we shall use the same letters $\chi_{\pm\pm}$ to denote one-dimensional representations of $G, G', G/G_0$, and G'/G'_0 .

Let $s \in S$ and $t \in T$. Since G' normalizes G'_0 , the quotient group G'/G'_0 acts naturally on

$$V(s, t) := \text{Diff}_{G'_0}(I(s), J(t)),$$

by $D \mapsto J(t)(g) \circ D \circ I(s)(g^{-1})$, and we have an irreducible decomposition:

$$V(s, t) \simeq \bigoplus_{\chi \in (G'/G'_0)^\wedge} V(s, t)_\chi$$

where $V(s, t)_\chi$ denotes the χ -component of $V(s, t)$. We note that

$$V(s, t)_\chi \simeq \text{Hom}_{G'}(I(s), J(t)) \quad (= \text{Hom}_{O(n, 1)}(I(i, \lambda)_\alpha, J(j, \nu)_\beta))$$

if $\chi = \chi_{++}$ (trivial representation).

We let the character group $(G/G_0)^\wedge$ act on S by the following formula:

$$\begin{aligned} \chi_{++} \cdot (i, \lambda, \alpha) &:= (i, \lambda, \alpha), & \chi_{+-} \cdot (i, \lambda, \alpha) &:= (i, \lambda, \alpha + 1), \\ \chi_{-+} \cdot (i, \lambda, \alpha) &:= (\tilde{i}, \lambda, \alpha + 1), & \chi_{--} \cdot (i, \lambda, \alpha) &:= (\tilde{i}, \lambda, \alpha). \end{aligned}$$

Then as in Lemma 2.2, we have a G -isomorphism

$$I(s) \otimes \chi \simeq I(\chi \cdot s) \quad \text{for any } \chi \in (G/G_0)^\wedge \text{ and } s \in S.$$

Therefore, we have natural isomorphisms as G'/G'_0 -modules:

$$\chi^{-1} \otimes \text{Diff}_{G'_0}(I(s), J(t)) \simeq \text{Diff}_{G'_0}(I(s) \otimes \chi, J(t)) \simeq \text{Diff}_{G'_0}(I(\chi \cdot s), J(t)).$$

Taking the χ_{++} -component of the both sides, we get an isomorphism

$$V(s, t)_\chi \simeq \text{Diff}_{G'}(I(\chi \cdot s), J(t)).$$

Thus we have proved a (G'/G'_0) -isomorphism:

$$V(s, t) \simeq \bigoplus_{\chi \in (G'/G'_0)^\wedge} \text{Diff}_{G'}(I(\chi \cdot s), J(t)).$$

There are four summands in the right-hand side, however, two of them vanish by the parity condition. In fact, if we take β as in the statement of the theorem, then the two summands for χ_{-+} and χ_{+-} vanish, as we shall see in Proposition 5.19 (1). Since $V(s, t) = \text{Hom}_{G'_0}(I(i, \lambda), J(j, \nu))$, the first equality in (2) has been proved. Likewise, we let the character group $(G'/G'_0)^\wedge$ act on the set T in a similar manner, as we did for S . Then we get a G' -isomorphism:

$$J(t) \otimes \chi \simeq J(\chi \cdot t) \quad \text{for any } \chi \in (G'/G'_0)^\wedge \text{ and } t \in T.$$

This leads us to the second equality. \square

2.6. Branching problems for Verma modules. In this section, we discuss briefly branching problems for generalized Verma modules for the pair

$$(\mathfrak{g}, \mathfrak{g}') = (\mathfrak{o}(n+2, \mathbb{C}), \mathfrak{o}(n+1, \mathbb{C})),$$

see [13] for the general problem. In [20, Thm. A] and [19], we established a duality theorem that gives a one-to-one correspondence between differential symmetry breaking operators and \mathfrak{g}' -homomorphisms for the restriction of Verma modules of \mathfrak{g} in the general setting, see Fact 3.3. Thus Theorem 2.8 for differential symmetry breaking operators leads us to the classification of \mathfrak{g}' -homomorphisms in certain branching problems of generalized Verma modules of \mathfrak{g} , and Theorem 2.9 constructs the corresponding “singular vectors”.

For a \mathfrak{p} -module F with trivial action of the nilpotent radical \mathfrak{n}_+ , we define a \mathfrak{g} -module (*generalized Verma module*) by

$$\text{ind}_{\mathfrak{p}}^{\mathfrak{g}}(F) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F.$$

If F is a P -module, then the \mathfrak{g} -module $\text{ind}_{\mathfrak{p}}^{\mathfrak{g}}(F)$ carries a P -module structure, and we may regard $\text{ind}_{\mathfrak{p}}^{\mathfrak{g}}(F)$ as a (\mathfrak{g}, P) -module.

We recall that $\sigma_{\lambda, \alpha}^{(i)}$ is a P -module whose restriction to $MA \simeq O(n) \times O(1) \times \mathbb{R}$ is given by $\bigwedge^i(\mathbb{C}^n) \boxtimes (-1)^\alpha \boxtimes \mathbb{C}_\lambda$ for $0 \leq i \leq n$, $\lambda \in \mathbb{C}$, $\alpha \in \mathbb{Z}/2\mathbb{Z}$. We set

$$\begin{aligned} M(i, \lambda)_\alpha &:= \text{ind}_{\mathfrak{p}}^{\mathfrak{g}}(\sigma_{\lambda, \alpha}^{(i)}) = \text{ind}_{\mathfrak{p}}^{\mathfrak{g}}(\bigwedge^i(\mathbb{C}^n) \boxtimes (-1)^\alpha \boxtimes \mathbb{C}_\lambda), \\ M(i, \lambda) &:= \text{ind}_{\mathfrak{p}}^{\mathfrak{g}}(\bigwedge^i(\mathbb{C}^n) \boxtimes \mathbb{C}_\lambda). \end{aligned}$$

Then $M(i, \lambda)_\alpha$ is a (\mathfrak{g}, P) -module, and $M(i, \lambda)$ is a \mathfrak{g} -module. The underlying \mathfrak{g} -module structure of $M(i, \lambda)_\alpha$ does not depend on $\alpha \in \mathbb{Z}/2\mathbb{Z}$, and we have the following isomorphisms as \mathfrak{g} -modules:

$$M(i, \lambda)_\alpha|_{\mathfrak{g}} \simeq M(i, \lambda) \simeq M(n - i, \lambda).$$

Similarly, for $0 \leq j \leq n - 1$, $\nu \in \mathbb{C}$, $\beta \in \mathbb{Z}/2\mathbb{Z}$, we set

$$\begin{aligned} M'(j, \nu)_\beta &:= \text{ind}_{\mathfrak{p}'}^{\mathfrak{g}'} \left(\tau_{\nu, \beta}^{(j)} \right) = \text{ind}_{\mathfrak{p}'}^{\mathfrak{g}'} \left(\bigwedge^j (\mathbb{C}^{n-1}) \boxtimes (-1)^\beta \boxtimes \mathbb{C}_\nu \right), \\ M'(j, \nu) &:= \text{ind}_{\mathfrak{p}'}^{\mathfrak{g}'} \left(\bigwedge^j (\mathbb{C}^{n-1}) \boxtimes \mathbb{C}_\nu \right). \end{aligned}$$

Then $M'(j, \nu)_\beta$ is a (\mathfrak{g}', P') -module and $M'(j, \nu)$ is a \mathfrak{g}' -module. We have the following isomorphisms as \mathfrak{g}' -modules.

$$M'(j, \nu)_\beta|_{\mathfrak{g}'} \simeq M'(j, \nu) \simeq M'(n - 1 - j, \nu).$$

As a part of branching problems, we wish to understand how the \mathfrak{g} -module $M(i, \lambda)$ behaves when restricted to the subalgebra \mathfrak{g}' , or how the (\mathfrak{g}, P) -module $M(i, P)_\alpha$ behaves as a (\mathfrak{g}', P') -module. As a dual to Theorem 2.10 (see Fact 3.3), we obtain:

Theorem 2.11. *Suppose $0 \leq i \leq n$, $0 \leq j \leq n - 1$, and $\lambda, \nu \in \mathbb{C}$.*

- (1) $\text{Hom}_{\mathfrak{g}'} (M'(\tilde{j}, -\nu), M(\tilde{i}, -\lambda)|_{\mathfrak{g}'}) \neq \{0\}$ only if $\nu - \lambda \in \mathbb{N}$.
- (2) Assume $\nu - \lambda \in \mathbb{N}$. We fix $\alpha \in \mathbb{Z}/2\mathbb{Z}$ and set $\beta := \alpha + \nu - \lambda \bmod 2$. Then we have

$$\begin{aligned} &\text{Hom}_{\mathfrak{g}'} (M'(\tilde{j}, -\nu), M(\tilde{i}, -\lambda)) \\ &\simeq \text{Hom}_{\mathfrak{g}', P'} (M'(\tilde{j}, -\nu)_\beta, M(\tilde{i}, -\lambda)_\alpha) \bigoplus \text{Hom}_{\mathfrak{g}', P'} (M'(\tilde{j}, -\nu)_\beta, M(i, -\lambda)_\alpha) \\ &\simeq \text{Hom}_{\mathfrak{g}', P'} (M'(\tilde{j}, -\nu)_\beta, M(\tilde{i}, -\lambda)_\alpha) \bigoplus \text{Hom}_{\mathfrak{g}', P'} (M'(j, -\nu)_\beta, M(\tilde{i}, -\lambda)_\alpha). \end{aligned}$$

The summands in the right-hand sides in (2) of Theorem 2.11 are classified as follows.

Proposition 2.12. *Let $n \geq 3$. Suppose $0 \leq i \leq n$, $0 \leq j \leq n - 1$, $\lambda, \nu \in \mathbb{C}$, and $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$. Then the following three conditions on 6-tuple $(i, j, \lambda, \nu, \alpha, \beta)$ are equivalent:*

- (i) $\text{Hom}_{\mathfrak{g}', P'} (M'(\tilde{j}, -\nu)_\beta, M(\tilde{i}, -\lambda)_\alpha) \neq \{0\}$.
- (ii) $\dim \text{Hom}_{\mathfrak{g}', P'} (M'(\tilde{j}, -\nu)_\beta, M(\tilde{i}, -\lambda)_\alpha) = 1$.
- (iii) The 6-tuple $(i, j, \lambda, \nu, \alpha, \beta)$ belongs to one of the six cases in Theorem 2.8 (iii).

The left-hand side of the isomorphisms in Theorem 2.11 is isomorphic to

$$\text{Hom}_{\mathfrak{p}'} \left(\bigwedge^{n-1-j} (\mathbb{C}^{n-1}) \otimes \mathbb{C}_{-\nu}, \text{ind}_{\mathfrak{p}}^{\mathfrak{g}} \left(\bigwedge^{n-i} (\mathbb{C}^n) \otimes \mathbb{C}_{-\lambda} \right) \right)$$

and vectors in the image of \mathfrak{p}' -homomorphisms are sometimes referred to as singular vectors. Fact 3.3 in the next chapter asserts that one could get one from another among the following:

- (explicit construction of) singular vectors;
- (explicit construction of) symmetry breaking operators (Theorem 2.9);
- (explicit construction of) polynomial solutions to the F-system (Theorems 6.1 and 7.3).

3. F-METHOD FOR MATRIX-VALUED DIFFERENTIAL OPERATORS

In this chapter we recall from [13, 14, 19, 20] a method based on the Fourier transform (*F-method*) to find explicit formulæ of differential symmetry breaking operators. For our purpose we need to develop the F-method for matrix-valued operators. A new ingredient is a canonical decomposition of the algebraic Fourier transform of the vector-valued principal series representations into a “scalar part” involving differential operators of higher order and into a “vector part” of first order. This is formulated and proved in Section 3.4.

3.1. Algebraic Fourier transform. Let E be a vector space over \mathbb{C} . The Weyl algebra $\mathcal{D}(E)$ is the ring of holomorphic differential operators on E with polynomial coefficients.

Definition 3.1. We define the *algebraic Fourier transform* as an algebra isomorphism of two Weyl algebras on E and its dual space E^\vee :

$$\mathcal{D}(E) \rightarrow \mathcal{D}(E^\vee), \quad T \mapsto \widehat{T},$$

induced by

$$(3.1) \quad \widehat{\frac{\partial}{\partial z_\ell}} := -\zeta_\ell, \quad \widehat{z}_\ell := \frac{\partial}{\partial \zeta_\ell}, \quad 1 \leq \ell \leq n,$$

where $n = \dim_{\mathbb{C}} E$, (z_1, \dots, z_n) are coordinates on E and $(\zeta_1, \dots, \zeta_n)$ are the dual coordinates on E^\vee .

Any linear transformation $A \in GL(E)$ gives rise to bijections

$$\begin{aligned} A_\# : \text{Pol}(E) &\longrightarrow \text{Pol}(E), & F &\mapsto F(A^{-1}\cdot), \\ A_* : \mathcal{D}(E) &\longrightarrow \mathcal{D}(E), & T &\mapsto A_\# \circ T \circ A_\#^{-1}. \end{aligned}$$

We write ${}^tA \in GL(E^\vee)$ for the dual map. Then the following identity holds [20, Lem. 3.3]:

$$(3.2) \quad \widehat{A_* T} = ({}^tA^{-1})_* \widehat{T} \quad \text{for all } T \in \mathcal{D}(E).$$

3.2. Differential operators between two manifolds.

We need a generalized notion of differential operators, not only for functions on the *same* manifolds but also for functions on two *different* manifolds with a morphism.

Let $\mathcal{V} \rightarrow X$ be a vector bundle over a smooth manifold X . We write $C^\infty(X, \mathcal{V})$ for the space of smooth sections, endowed with the Fréchet topology of uniform convergence of sections and their derivatives of finite order on compact sets. Let $\mathcal{W} \rightarrow Y$ be another vector bundle. Suppose a smooth map $p: Y \rightarrow X$ is given.

Definition 3.2. ([20, Def. 2.1]) We say a continuous operator $T: C^\infty(X, \mathcal{V}) \rightarrow C^\infty(Y, \mathcal{W})$ is a *differential operator* if T satisfies

$$p(\text{Supp} Tf) \subset \text{Supp} f \quad \text{for all } f \in C^\infty(X, \mathcal{V}).$$

We write $\text{Diff}(\mathcal{V}_X, \mathcal{W}_Y)$ for the space of differential operators from $C^\infty(X, \mathcal{V})$ to $C^\infty(Y, \mathcal{W})$.

If $i: Y \rightarrow X$ is an immersion, then every $T \in \text{Diff}(\mathcal{V}_X, \mathcal{W}_Y)$ is locally of the form

$$T = \sum_{\alpha \in \mathbb{N}^k} \sum_{\beta \in \mathbb{N}^m} g_{\alpha, \beta}(y) \frac{\partial^{|\alpha|+|\beta|}}{\partial y^\alpha \partial z^\beta} \quad (\text{finite sum}),$$

where $(y_1, \dots, y_k, z_1, \dots, z_m)$ are local coordinates on X such that Y is given by $z_1 = \dots = z_m = 0$ and $g_{\alpha, \beta}(y)$ are $\text{Hom}(V, W)$ -valued smooth functions on Y .

3.3. F-method for principal series representations.

Let G be a real reductive Lie group, and $P = MAN_+$ a Langlands decomposition of a parabolic subgroup P of G . Their Lie algebras will be denoted by $\mathfrak{g}(\mathbb{R})$, $\mathfrak{p}(\mathbb{R}) = \mathfrak{m}(\mathbb{R}) + \mathfrak{a}(\mathbb{R}) + \mathfrak{n}_+(\mathbb{R})$, and the complexified Lie algebras by \mathfrak{g} , $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}_+$, respectively.

Given $\lambda \in \mathfrak{a}^* \simeq \text{Hom}_{\mathbb{R}}(\mathfrak{a}(\mathbb{R}), \mathbb{C})$, we define one-dimensional representation \mathbb{C}_λ of A by $a \mapsto a^\lambda := e^{\langle \lambda, \log a \rangle}$. By letting MN_+ act trivially, we also regard \mathbb{C}_λ as a representation of P . Given a representation (σ, V) of M and $\lambda \in \mathfrak{a}^*$, we write $\sigma_\lambda \equiv \sigma \boxtimes \mathbb{C}_\lambda$ for the representation of MA on V defined by $ma \mapsto a^\lambda \sigma(m)$. The same letter will be used for the representation of P which is obtained by letting N_+ act trivially. We define $\mathcal{V} \equiv \mathcal{V}_X = G \times_P V$ as a G -equivariant vector bundle over the real flag variety $X = G/P$ associated to σ_λ . The (unnormalized) principal series representation $\pi_{(\sigma, \lambda)} = \text{Ind}_P^G(\sigma_\lambda)$ is defined on the Fréchet space $C^\infty(X, \mathcal{V})$ of smooth sections of the vector bundle $\mathcal{V} \rightarrow X$.

Let $\mathfrak{g}(\mathbb{R}) = \mathfrak{n}_-(\mathbb{R}) + \mathfrak{m}(\mathbb{R}) + \mathfrak{a}(\mathbb{R}) + \mathfrak{n}_+(\mathbb{R})$ be the Gelfand–Naimark decomposition. The vector bundle $\mathcal{V} \rightarrow X$ is trivialized when restricted to the open Bruhat cell

$$\mathfrak{n}_-(\mathbb{R}) \simeq N_- \hookrightarrow G/P = X,$$

and we may regard $C^\infty(X, \mathcal{V})$ as a subspace of $C^\infty(\mathfrak{n}_-(\mathbb{R})) \otimes V$ via the restriction. This model is called the *N-picture* or *flat picture* of the principal series representation and the case of the Lorentz group $G = O(n+1, 1)$ was discussed in detail in Chapter 2. The infinitesimal representation of the Lie algebra on $C^\infty(\mathfrak{n}_-(\mathbb{R})) \otimes V$ will be denoted by $d\pi_{(\sigma, \lambda)}$.

Let $2\rho \in \mathfrak{a}^*$ be the homomorphism on $\mathfrak{a}(\mathbb{R})$ defined by $Z \mapsto \text{Trace}(\text{ad}(Z)): \mathfrak{n}_+(\mathbb{R}) \rightarrow \mathfrak{n}_+(\mathbb{R})$. As a representation of P , $\mathbb{C}_{2\rho}$ is given by $p \mapsto \chi_{2\rho}(p) := |\det(\text{Ad}(p): \mathfrak{n}_+(\mathbb{R}) \rightarrow \mathfrak{n}_+(\mathbb{R}))|$. We also define a one-dimensional representation sgn of P by $p \mapsto \text{sgn} \circ$

$\det(\mathrm{Ad}(p): \mathfrak{n}_+(\mathbb{R}) \rightarrow \mathfrak{n}_+(\mathbb{R}))$. Observe that the density bundle Ω_X and the orientation bundle of $X = G/P$ are then given as the homogeneous line bundles $G \times_P \mathbb{C}_{2\rho}$ and $G \times_P \mathrm{sgn}$, respectively. Since MN_+ acts trivially on $\mathbb{C}_{2\rho}$, we shall sometimes regard $\mathbb{C}_{2\rho}$ as a representation of $A \simeq P/MN_+$. Write $V^\vee = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and let (σ^\vee, V^\vee) denote the contragredient representation of the finite-dimensional representation (σ, V) of M . For $\lambda \in \mathfrak{a}^*$, we define two representations of P by $(\sigma_\lambda)^\vee := (\sigma^\vee)_{-\lambda} = \sigma^\vee \boxtimes \mathbb{C}_{-\lambda}$ and $\sigma_\lambda^* := \sigma^\vee \boxtimes \mathbb{C}_{2\rho-\lambda}$ with trivial action of N_+ as before, and form a representation

$$\pi_{(\sigma, \lambda)^*} = \mathrm{Ind}_P^G(\sigma_\lambda^*)$$

of G on $C^\infty(X, \mathcal{V}^*)$ where $\mathcal{V}^* = G \times_P V^\vee$ is the dualizing bundle associated to the representation σ_λ^* of P . The integration over X gives rise to a natural G -invariant nondegenerate bilinear form

$$\mathrm{Ind}_P^G(\sigma_\lambda) \times \mathrm{Ind}_P^G(\sigma_\lambda^*) \longrightarrow \mathbb{C}.$$

The infinitesimal representation of $\pi_{(\sigma, \lambda)^*}$ in the N -picture is given by a Lie algebra homomorphism

$$d\pi_{(\sigma, \lambda)^*}: \mathfrak{g} \longrightarrow \mathcal{D}(\mathfrak{n}_-) \otimes \mathrm{End}(V^\vee).$$

Applying the algebraic Fourier transform of the Weyl algebra (see Definition 3.1), we get a Lie algebra homomorphism

$$\widehat{d\pi_{(\sigma, \lambda)^*}}: \mathfrak{g} \longrightarrow \mathcal{D}(\mathfrak{n}_+) \otimes \mathrm{End}(V^\vee),$$

where we have identified \mathfrak{n}_-^\vee with \mathfrak{n}_+ by an $\mathrm{Ad}(G)$ -invariant, nondegenerate symmetric bilinear form on \mathfrak{g} .

We define a \mathfrak{g} -module (*generalized Verma module*) by

$$\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V^\vee,$$

where V^\vee is regarded as a \mathfrak{p} -module through $d\sigma^\vee \otimes (-\lambda)$ with trivial \mathfrak{n}_+ -action. We let P act on V^\vee by $(\sigma_\lambda)^\vee$. Then the \mathfrak{g} -module $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ carries a P -module structure, so that we may regard $\mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ as a (\mathfrak{g}, P) -module. This observation will be useful when G is a real reductive Lie group because the parabolic subgroup P may be disconnected. We recall from [20, (3.23)] that the algebraic Fourier transform of the generalized Verma module is a (\mathfrak{g}, P) -isomorphism

$$F_c: \mathrm{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee) \xrightarrow{\sim} \mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee,$$

where $\mathrm{Pol}(\mathfrak{n}_+) \otimes V^\vee$ is regarded as a (\mathfrak{g}, P) -module via $\widehat{d\pi_{(\sigma, \lambda)^*}}$.

Let G' be a real reductive subgroup of G , and P' a parabolic subgroup of G' . Given a finite-dimensional representation W of P' , we define two homogeneous vector

bundles:

$$\begin{aligned}\mathcal{W}_Y &:= G' \times_{P'} W \longrightarrow Y := G'/P', \\ \mathcal{W}_Z &:= G \times_{P'} W \longrightarrow Z := G/P' .\end{aligned}$$

Similarly to the representation (σ_λ, V) of $P = MAN_+$, we shall consider a representation (τ_ν, W) of $P' = M'A'N'_+$ which extends the outer tensor product representation $\tau \boxtimes \mathbb{C}_\nu$ for $\nu \in (\mathfrak{a}')^*$ by letting N'_+ act trivially.

We note that the base space Z is not compact in general, whereas Y is a real flag variety and thus compact. If $P' \subset P$, then there are natural maps:

$$Y \longrightarrow Z \twoheadrightarrow X.$$

We denote by $\text{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y)$ the space of G' -equivariant operators from $C^\infty(X, \mathcal{V}_X)$ to $C^\infty(Y, \mathcal{W}_Y)$ which are differential operators with respect to the above G' -equivariant map $Y \rightarrow X$ in the sense of Definition 3.2. The space $\text{Diff}_G(\mathcal{V}_X, \mathcal{W}_Z)$ is defined in a similar way.

The map taking symbols of differential operators on \mathbb{R}^n , to be denoted by Symb , induces an isomorphism below when restricted to differential operators with constant coefficients,

$$(3.3) \quad \text{Symb} : \text{Diff}^{\text{const}}(C^\infty(\mathbb{R}^n) \otimes V, C^\infty(\mathbb{R}^n) \otimes W) \xrightarrow{\sim} \text{Pol}[\zeta_1, \dots, \zeta_n] \otimes \text{Hom}_{\mathbb{C}}(V, W)$$

such that

$$e^{-\langle z, \zeta \rangle} D(e^{\langle z, \zeta \rangle} \otimes v) = \text{Symb}(D)(v) \in \text{Pol}[\zeta_1, \dots, \zeta_n] \otimes W$$

for all $v \in V$. We summarize the F-method in this setting from [20, Thm. 2.9, Rem. 2.18, Thm. 4.1, Cor. 4.3]:

Fact 3.3. *Let $G \supset G'$ be a pair of real reductive Lie groups, and $P \supset P'$ a pair of parabolic subgroups with compatible Levi decompositions $P = LN_+ \supset P' = L'N'_+$ such that $L \supset L'$ and $N_+ \supset N'_+$. Let (σ_λ, V) and (τ_ν, W) be finite-dimensional representations of P and P' with trivial actions of N_+ and N'_+ , respectively.*

(1) (duality) *There is a natural isomorphism:*

$$D_{X \rightarrow Y} : \text{Hom}_{\mathfrak{g}', P'}(\text{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee), \text{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) \xrightarrow{\sim} \text{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y).$$

(2) (extension) *The restriction $\mathcal{W}_Z|_Y \simeq \mathcal{W}_Y$ induces the bijection*

$$\text{Rest}_Y : \text{Diff}_G(\mathcal{V}_X, \mathcal{W}_Z) \xrightarrow{\sim} \text{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y).$$

(3) (F-method) *For $\psi \in (\text{Pol}(\mathfrak{n}_+) \otimes V^\vee) \otimes W \simeq \text{Hom}_{\mathbb{C}}(V, W \otimes \text{Pol}(\mathfrak{n}_+))$, we consider a system of partial differential equations (F-system)*

$$(3.4) \quad \widehat{(d\pi_{(\sigma, \lambda)}^*(C) \otimes \text{id}_W)} \psi = 0 \quad \text{for all } C \in \mathfrak{n}'_+,$$

and set

$$(3.5) \quad \text{Sol}(\mathfrak{n}_+; \sigma_\lambda, \tau_\nu) := \{\psi \in \text{Hom}_{L'}(V, W \otimes \text{Pol}(\mathfrak{n}_+)) : \psi \text{ solves (3.4)}\}.$$

Then there is a natural isomorphism

$$(3.6) \quad \text{Hom}_{\mathfrak{g}', P'}(\text{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee), \text{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) \xrightarrow{\sim} \text{Sol}(\mathfrak{n}_+; \sigma_\lambda, \tau_\nu).$$

- (4) Assume that the nilradical \mathfrak{n}_+ is abelian. Then, the system (3.4) is of second order, and the following diagram of six isomorphisms commutes:

$$\begin{array}{ccccc}
 & & \text{Sol}(\mathfrak{n}_+; \sigma_\lambda, \tau_\nu) & & \\
 & \nearrow^{F_c \otimes \text{id}} & \uparrow & \nwarrow^{\text{Rest}_Y \circ \text{Symb}^{-1}} & \\
 & & \text{Diff}_G(\mathcal{V}_X, \mathcal{W}_Z) & & \\
 \text{Hom}_{\mathfrak{g}', P'}(\text{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee), \text{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) & \xrightarrow{D_{X \rightarrow Z}} & & \xrightarrow{\text{Rest}_Y} & \text{Diff}_{G'}(\mathcal{V}_X, \mathcal{W}_Y) \\
 & \searrow_{D_{X \rightarrow Y}} & & &
 \end{array}$$

Fact 3.3 (3) implies that, once we find such a polynomial solution ψ to the F-system, we obtain a P' -submodule W^\vee in $\text{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)$ (sometimes referred to as *singular vectors*) by $(F_c \otimes \text{id})^{-1}(\psi)$, where we have used the canonical isomorphism $\text{Hom}_{P'}(W^\vee, \text{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee)) \simeq \text{Hom}_{\mathfrak{g}', P'}(\text{ind}_{\mathfrak{p}'}^{\mathfrak{g}'}(W^\vee), \text{ind}_{\mathfrak{p}}^{\mathfrak{g}}(V^\vee))$ when we apply the algebraic Fourier transform F_c of a generalized Verma module. Simultaneously, we obtain a differential symmetry breaking operator by $\text{Rest}_Y \circ \text{Symb}^{-1}(\psi)$ in the flat picture (N -picture), when \mathfrak{n}_+ is abelian.

The following useful lemma guarantees that the F-system (3.4) can be verified by a single nonzero element $C \in \mathfrak{n}'_+$ when L' acts irreducibly on \mathfrak{n}'_+ , equivalently, when \mathfrak{n}'_+ is abelian.

Lemma 3.4. *Suppose \mathfrak{n}'_+ is abelian. Then the following two conditions on $\psi \in \text{Hom}_{L'}(V, \text{Pol}(\mathfrak{n}_+) \otimes W)$ are equivalent.*

- (i) *For every $C \in \mathfrak{n}'_+$, $\left(\widehat{d\pi_{(\sigma, \lambda)^*}}(C) \otimes \text{id}_W\right) \psi = 0$.*
- (ii) *For some nonzero $C_0 \in \mathfrak{n}'_+$, $\left(\widehat{d\pi_{(\sigma, \lambda)^*}}(C_0) \otimes \text{id}_W\right) \psi = 0$.*

Proof. The implication (i) \Rightarrow (ii) is obvious. We shall prove (ii) \Rightarrow (i). We set

$$\mu := \sigma_\lambda^*.$$

Suppose $\psi \in \text{Hom}_{L'}(V, \text{Pol}(\mathfrak{n}_+) \otimes W) \simeq (V^\vee \otimes \text{Pol}(\mathfrak{n}_+) \otimes W)^{L'}$. This means that

$$\chi_{2\rho}(\ell) \mu(\ell^{-1}) \otimes \text{Ad}_\#(\ell^{-1}) \otimes \tau_\nu(\ell^{-1}) \psi = \psi \quad \text{for all } \ell \in L'.$$

If ψ satisfies (ii), then we have

$$(3.7) \quad \left(\widehat{d\pi_\mu}(C_0) \otimes \text{id}_W \right) (\mu(\ell^{-1})\text{Ad}_\#(\ell^{-1})) \psi = 0.$$

We let the group L act on $V^\vee \otimes \text{Pol}(\mathfrak{n}_-)$ by $\pi_\mu(\ell) = \mu(\ell)\text{Ad}_\#(\ell)$. Then we have

$$d\pi_\mu(\text{Ad}(\ell)C_0) = \pi_\mu(\ell)d\pi_\mu(C_0)\pi_\mu(\ell^{-1}) \quad \text{for all } \ell \in L'(\subset L).$$

Applying (3.2) to the case where $E = \mathfrak{n}_-$, $A = \text{Ad}(\ell)$, and $T = \widehat{d\pi_\mu}(C_0)$, we have

$$\widehat{d\pi_\mu}(\text{Ad}(\ell)C_0) = \mu(\ell)\text{Ad}(\ell)_\# \widehat{d\pi_\mu}(C_0)\text{Ad}(\ell^{-1})_\# \mu(\ell^{-1}),$$

where we identify the action ${}^t\text{Ad}(\ell)^{-1}$ on \mathfrak{n}_-^\vee with the one of $\text{Ad}(\ell)$ on \mathfrak{n}_+ . Then

$$\left(\widehat{d\pi_\mu}(\text{Ad}(\ell)C_0) \otimes \text{id}_W \right) \psi = 0,$$

by (3.7). Since \mathfrak{n}'_+ is abelian, the Levi subgroup L' acts irreducibly on the nilradical \mathfrak{n}'_+ of the parabolic subalgebra $\mathfrak{p}'_+ = \mathfrak{l}' + \mathfrak{n}'_+$, and therefore $\text{Ad}(\ell)C_0$ ($\ell \in L'$) spans \mathfrak{n}'_+ . Hence (ii) \Rightarrow (i) is proved. \square

3.4. Matrix-valued differential operators in the F-method. This section provides a structural result on the key operator $\widehat{d\pi_{(\sigma,\lambda)}^*}$ in the F-method for the principal series representation $\text{Ind}_P^G(\sigma_\lambda)$ when P is a parabolic subgroup with abelian unipotent radical. We shall prove that $\widehat{d\pi_{(\sigma,\lambda)}^*}$ has a canonical decomposition into a sum of the “scalar part” (differential operator of second order) depending only on the continuous parameter $\lambda \in \mathfrak{a}^*$ and the “vector part” (differential operator of first order) depending only on $\sigma \in \widehat{M}$.

We retain the notation in Section 3.3, and simply write

$$\widehat{d\pi_{\lambda^*}}: \mathfrak{g} \longrightarrow \mathcal{D}(\mathfrak{n}_+),$$

for $\widehat{d\pi_{(\sigma,\lambda)}^*}$ when (σ, V) is the trivial one-dimensional representation. We define the “vector part” of $\widehat{d\pi_{(\sigma,\lambda)}^*}$ as a linear map $A_\sigma: \mathfrak{g} \longrightarrow \mathcal{D}(\mathfrak{n}_+) \otimes \text{End}(V^\vee)$ characterized by the formula

$$(3.8) \quad \widehat{d\pi_{(\sigma,\lambda)}^*}(Y) = \widehat{d\pi_{\lambda^*}}(Y) \otimes \text{id}_{V^\vee} + A_\sigma(Y) \quad \text{for } Y \in \mathfrak{g}.$$

Let $\{N_\ell^-\}$ be a basis of $\mathfrak{n}_-(\mathbb{R})$, and $(\zeta_1, \dots, \zeta_n)$ be the corresponding coordinates on $\mathfrak{n}_-^\vee(\mathbb{R}) \simeq \mathfrak{n}_+(\mathbb{R})$.

Proposition 3.5. *Assume \mathfrak{n}_+ is abelian. Then, for any $Y \in \mathfrak{n}_+$, $A_\sigma(Y)$ is a holomorphic vector field on \mathfrak{n}_+ with constant coefficients in $\text{End}(V^\vee)$. An explicit formula is given as follows.*

$$(3.9) \quad A_\sigma(Y)F = - \sum_{\ell=1}^n \frac{\partial}{\partial \zeta_\ell} F \circ d\sigma \left([Y, N_\ell^-] \Big|_{\mathfrak{m}} \right) \quad \text{for } F \in \text{Pol}(\mathfrak{n}_+) \otimes V^\vee.$$

In particular, the “vector part” A_σ is independent of the continuous parameter λ . Moreover, A_σ is zero if $d\sigma = 0$.

Proof. Let $G_{\mathbb{C}}$ be a connected complex Lie group with Lie algebra $\mathfrak{g} = \mathfrak{g}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$, and $P_{\mathbb{C}} = L_{\mathbb{C}} \exp \mathfrak{n}_+$ the parabolic subgroup with Lie algebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}_+$. According to the Gelfand–Naimark decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{l} + \mathfrak{n}_+$ of the Lie algebra \mathfrak{g} , we have a diffeomorphism

$$\mathfrak{n}_- \times L_{\mathbb{C}} \times \mathfrak{n}_+ \rightarrow G_{\mathbb{C}}, \quad (Z, \ell, Y) \mapsto (\exp Z)\ell(\exp Y),$$

into an open dense subset $G_{\mathbb{C}}^{\text{reg}}$ of $G_{\mathbb{C}}$. Let

$$p_{\pm}: G_{\mathbb{C}}^{\text{reg}} \longrightarrow \mathfrak{n}_{\pm}, \quad p_o: G_{\mathbb{C}}^{\text{reg}} \rightarrow L_{\mathbb{C}},$$

be the projections characterized by the identity

$$\exp(p_-(g))p_o(g)\exp(p_+(g)) = g.$$

Then the following maps α and β are determined by the Gelfand–Naimark decomposition $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{l} + \mathfrak{n}_+$ and independent of the choice of the complex Lie group $G_{\mathbb{C}}$:

$$(3.10) \quad (\alpha, \beta): \mathfrak{g} \times \mathfrak{n}_- \rightarrow \mathfrak{l} \oplus \mathfrak{n}_-, \quad (Y, Z) \mapsto \left. \frac{d}{dt} \right|_{t=0} (p_o(e^{tY}e^Z), p_-(e^{tY}e^Z)).$$

According to the direct sum decomposition $\mathfrak{l} = \mathfrak{m} + \mathfrak{a}$, we write

$$\alpha(Y, Z) = \alpha(Y, Z)|_{\mathfrak{m}} + \alpha(Y, Z)|_{\mathfrak{a}}.$$

For a fixed element $Y \in \mathfrak{g}$, $\beta(Y, \cdot)$ induces a complex linear map $\mathfrak{n}_- \rightarrow \mathfrak{n}_-$, and thus we may regard $\beta(Y, \cdot)$ as a holomorphic vector field on \mathfrak{n}_- via the identification of \mathfrak{n}_- with the holomorphic tangent space at each point:

$$\mathfrak{n}_- \ni Z \mapsto \beta(Y, Z) \in \mathfrak{n}_- \simeq T_Z \mathfrak{n}_-.$$

Suppose $f \in C^\infty(\mathfrak{n}_-(\mathbb{R}), V^\vee)$, $Y \in \mathfrak{g}(\mathbb{R})$ and $Z \in \mathfrak{n}_-(\mathbb{R})$. Since N_+ acts trivially on V , the infinitesimal representation $d\pi_{(\sigma, \lambda)^*}$ is given by

$$d\pi_{(\sigma, \lambda)^*}(Y)f(Z) = d\sigma^\vee(\alpha(Y, Z)|_{\mathfrak{m}})f(Z) + (d\lambda^*(\alpha(Y, Z)|_{\mathfrak{a}})f(Z) - \beta(Y, \cdot)f(Z)).$$

In view of the decomposition, we define $d\pi_{(\sigma, \lambda)^*}^{\text{vect}}, d\pi_{(\sigma, \lambda)^*}^{\text{scalar}} \in \text{Hom}_{\mathbb{C}}(\mathfrak{g}, \mathcal{D}(\mathfrak{n}_-) \otimes \text{End}(V^\vee))$ by

$$\begin{aligned} d\pi_{(\sigma, \lambda)^*}^{\text{vect}}(Y) &:= d\sigma^\vee(\alpha(Y, Z)|_{\mathfrak{m}}), \\ d\pi_{(\sigma, \lambda)^*}^{\text{scalar}}(Y) &:= d\lambda^*(\alpha(Y, Z)|_{\mathfrak{a}}) - \beta(Y, \cdot). \end{aligned}$$

Clearly, $d\pi_{(\sigma,\lambda)^*} = d\pi_{(\sigma,\lambda)^*}^{\text{scalar}} + d\pi_{(\sigma,\lambda)^*}^{\text{vect}}$. We say $d\pi_{(\sigma,\lambda)^*}^{\text{scalar}}$ is the *scalar part* of $d\pi_{(\sigma,\lambda)^*}$, and $d\pi_{(\sigma,\lambda)^*}^{\text{vect}}$ is the *vector part*. Then the scalar part $d\pi_{(\sigma,\lambda)^*}^{\text{scalar}}$ does not depend on (σ, V) , and takes the form

$$d\pi_{(\sigma,\lambda)^*}^{\text{scalar}}(Y) = d\pi_{\lambda^*}(Y) \otimes \text{id}_{V^\vee} \quad \text{for all } Y \in \mathfrak{g}.$$

Let us compute their algebraic Fourier transforms. Obviously, the algebraic Fourier transform of $d\pi_{(\sigma,\lambda)^*}^{\text{scalar}}(Y)$ is $\widehat{d\pi_{\lambda^*}(Y) \otimes \text{id}_{V^\vee}}$.

If \mathfrak{n}_+ is abelian, we have

$$\alpha(Y, Z) = [Y, Z], \quad \beta(Y, Z) = \frac{1}{2}[Z, [Z, Y]] \quad \text{for } Y \in \mathfrak{n}_+ \text{ and } Z \in \mathfrak{n}_-,$$

see [20, Lem. 3.8].

We write $Z = \sum_{\ell} z_{\ell} N_{\ell}^-$. Then for $Y \in \mathfrak{n}_+$, we have

$$\begin{aligned} (d\pi_{(\sigma,\lambda)^*}^{\text{vect}}(Y)f)(Z) &= -f(Z) \circ d\sigma([Y, Z]|_{\mathfrak{m}}) \\ &= -\sum_{\ell} z_{\ell} f \circ d\sigma([Y, N_{\ell}^-]|_{\mathfrak{m}}). \end{aligned}$$

By (3.1) its algebraic Fourier transform is given by (3.9). The remaining assertions of Proposition 3.5 are clear. \square

4. MATRIX-VALUED F-METHOD FOR $O(n+1, 1)$

This chapter summarizes a strategy and technical details in applying the F-method to find matrix-valued symmetry breaking operators in the setting where $(G, G') = (O(n+1, 1), O(n, 1))$.

4.1. Strategy of matrix-valued F-method for $(G, G') = (O(n+1, 1), O(n, 1))$. We retain the notation of Chapter 2. In particular, $P = L \exp(\mathfrak{n}_+(\mathbb{R}))$ and $P' = L' \exp(\mathfrak{n}'_+(\mathbb{R}))$ are the minimal parabolic subgroups of $G = O(n+1, 1)$ and $G' = O(n, 1)$, respectively, such that $L \supset L'$ and $\mathfrak{n}_+(\mathbb{R}) \supset \mathfrak{n}'_+(\mathbb{R})$. We recall $L = MA \simeq O(n) \times O(1) \times \mathbb{R}$ and $\mathfrak{n}_\pm(\mathbb{R})$ is identified with \mathbb{R}^n via the basis $\{N_1^\pm, \dots, N_n^\pm\}$, see (2.2). Let $(\zeta_1, \dots, \zeta_n)$ be the coordinates of $\mathfrak{n}_+(\simeq \mathfrak{n}_-^\vee)$. Then the L -module $\text{Pol}(\mathfrak{n}_+)$ is identified with the polynomial ring $\text{Pol}[\zeta_1, \dots, \zeta_n]$ on which the action of $L = MA \ni ((B, b), e^{tH_0})$ is given by

$$(4.1) \quad f(\zeta) \mapsto f(b^{-1}e^{-t}B^{-1}\zeta) \quad \text{for } \zeta = {}^t(\zeta_1, \dots, \zeta_n).$$

The subgroup $L' = M'A \simeq O(n-1) \times O(1) \times \mathbb{R}$ stabilizes the last variable ζ_n , and acts irreducibly on $\mathfrak{n}'_+ \simeq \mathbb{C}^{n-1}$. Then we may apply Lemma 3.4 by choosing $C_0 = N_1^+$. With this notation, the F -method (Fact 3.3) implies the following:

Proposition 4.1. *Let $(G, G') = (O(n+1, 1), O(n, 1))$, $\sigma_\lambda = \sigma \boxtimes \mathbb{C}_\lambda$ be a finite-dimensional representation of P on V that factors the quotient group $P/N_+ \simeq L = MA$, and $\tau_\nu = \tau \boxtimes \mathbb{C}_\nu$ be that of P' that factors $P'/N'_+ \simeq L' = M'A$ on W . The flat pictures of the principal series representations $\text{Ind}_P^G(\sigma_\lambda)$ of G and $\text{Ind}_{P'}^{G'}(\tau_\nu)$ of G' are defined in $C^\infty(\mathbb{R}^n) \otimes V$ and $C^\infty(\mathbb{R}^{n-1}) \otimes W$, respectively, as in (2.7). Then we have the following.*

- (1) $\text{Sol}(\mathfrak{n}_+; \sigma_\lambda, \tau_\nu)$ (see (3.5)) is given by

$$(4.2) \quad \begin{aligned} & \text{Sol}(\mathfrak{n}_+; \sigma_\lambda, \tau_\nu) \\ &= \left\{ \psi \in \text{Hom}_{L'}(V, W \otimes \text{Pol}[\zeta_1, \dots, \zeta_n]) : \left(\widehat{d\pi_{(\sigma, \lambda)^*}(N_1^+)} \otimes \text{id}_W \right) \psi = 0 \right\}. \end{aligned}$$

- (2) Suppose $\psi \in \text{Sol}(\mathfrak{n}_+; \sigma_\lambda, \tau_\nu)$. Let D be the $\text{Hom}_{\mathbb{C}}(V, W)$ -valued differential operator on \mathbb{R}^n with constant coefficients such that $\text{Symb}(D) = \psi$. Then the differential operator

$$\text{Rest}_{x_n=0} \circ D : C^\infty(\mathbb{R}^n) \otimes V \rightarrow C^\infty(\mathbb{R}^{n-1}) \otimes W$$

extends to a symmetry breaking operator from $\text{Ind}_P^G(\sigma_\lambda)$ to $\text{Ind}_{P'}^{G'}(\tau_\nu)$.

- (3) Conversely, any G' -equivariant differential operator from $\text{Ind}_P^G(\sigma_\lambda)$ to $\text{Ind}_{P'}^{G'}(\tau_\nu)$ is obtained in this manner.

The rest of this chapter is devoted to an explicit characterization of the main ingredients of Proposition 4.1. Namely, the space $\text{Hom}_{L'}(V, W \otimes \text{Pol}[\zeta_1, \dots, \zeta_n])$ is

described in Section 4.3, the scalar and vector parts of the operator $\widehat{d\pi_{(\sigma,\lambda)}^*}(N_1^+)$ are given in Section 4.4 and the matrix components of (4.2) are studied in Section 4.5.

Harmonic polynomials play a key role in the first two steps of this characterization.

4.2. Harmonic polynomials.

We review a classical fact on harmonic polynomials. Let $N \in \mathbb{N}_+$ (later, we take N to be $n-1$ or n). For $k \in \mathbb{N}$, we denote by $\text{Pol}^k[\zeta_1, \dots, \zeta_N]$ the space of homogeneous polynomials of degree k . The space $\mathcal{H}^k(\mathbb{C}^N)$ of harmonic polynomials of degree k is defined by

$$\mathcal{H}^k(\mathbb{C}^N) := \{h \in \text{Pol}^k[\zeta_1, \dots, \zeta_N] : \Delta_{\mathbb{C}^N} h = 0\},$$

where $\Delta_{\mathbb{C}^N} := \frac{\partial^2}{\partial \zeta_1^2} + \dots + \frac{\partial^2}{\partial \zeta_N^2}$ denotes the holomorphic Laplacian on \mathbb{C}^N . Then $\mathcal{H}^k(\mathbb{C}^N) \neq \{0\}$ for all $k \in \mathbb{N}$ if $N \geq 2$ and $\mathcal{H}^k(\mathbb{C}^1) \neq \{0\}$ for $k \in \{0, 1\}$.

The orthogonal group $O(N)$ acts irreducibly on $\mathcal{H}^k(\mathbb{C}^N)$ for all $k \in \mathbb{N}$ unless it is zero. We set

$$\mathcal{H}(\mathbb{C}^N) := \bigoplus_{k=0}^{\infty} \mathcal{H}^k(\mathbb{C}^N).$$

Then we have a natural decomposition of the space of polynomials into spherical harmonics and $O(N)$ -invariant polynomials for any $N \in \mathbb{N}_+$:

$$(4.3) \quad \text{Pol}[\zeta_1^2 + \dots + \zeta_N^2] \otimes \mathcal{H}(\mathbb{C}^N) \xrightarrow{\sim} \text{Pol}[\zeta_1, \dots, \zeta_N].$$

4.3. Description of $\text{Hom}_{L'}(V, W \otimes \text{Pol}(\mathfrak{n}_+))$. As the first step of the matrix-valued F-method for the Lorentz group $G = O(n+1, 1)$, we give a description of the space $\text{Hom}_{L'}(V, W \otimes \text{Pol}(\mathfrak{n}_+))$ by using harmonic polynomials.

We retain the notation of Section 4.1, in particular, $(\zeta_1, \dots, \zeta_n)$ are the coordinates of \mathfrak{n}_+ such that \mathfrak{n}'_+ is characterized by $\zeta_n = 0$. For $b \in \mathbb{Z}$ and a polynomial $g(t)$ of one variable t , we define a multi-valued meromorphic function of n variables $\zeta = (\zeta_1, \dots, \zeta_n)$ by

$$(4.4) \quad (T_b g)(\zeta) := Q_{n-1}(\zeta')^{\frac{b}{2}} g\left(\frac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}\right),$$

where $\zeta' = (\zeta_1, \dots, \zeta_{n-1})$ and $Q_{n-1}(\zeta') = \zeta_1^2 + \dots + \zeta_{n-1}^2$. Clearly, $(T_b g)(\zeta) \equiv 0$ if and only if $g(t) \equiv 0$. We observe that $(T_b g)(\zeta)$ is a homogeneous polynomial of $(\zeta_1, \dots, \zeta_n)$ of degree b if $b \in \mathbb{N}$ and $g \in \text{Pol}_b[t]_{\text{even}}$, where we set

$$(4.5) \quad \text{Pol}_b[t]_{\text{even}} := \mathbb{C}\text{-span} \left\langle t^{b-2j} : 0 \leq j \leq \left\lfloor \frac{b}{2} \right\rfloor \right\rangle.$$

Then we have the following bijection

$$(4.6) \quad T_b: \text{Pol}_b[t]_{\text{even}} \xrightarrow{\sim} \bigoplus_{2\ell+c=b} \text{Pol}^\ell[\zeta_1^2 + \cdots + \zeta_{n-1}^2] \otimes \text{Pol}^c[\zeta_n].$$

Lemma 4.2. *Suppose $n \geq 2$. Then for every $a \in \mathbb{N}$, there is a natural bijection:*

$$\bigoplus_{k=0}^a \text{Pol}_{a-k}[t]_{\text{even}} \otimes \text{Hom}_{O(n-1)}(V, W \otimes \mathcal{H}^k(\mathbb{C}^{n-1})) \xrightarrow{\sim} \text{Hom}_{O(n-1)}(V, W \otimes \text{Pol}^a(\mathfrak{n}_+))$$

induced by

$$\sum_{k=0}^a g_k \otimes H^{(k)} \mapsto \sum_{k=0}^a (T_{a-k} g_k) H^{(k)}.$$

Proof. Combining the following two $O(n-1)$ -isomorphisms

$$\text{Pol}^a(\mathfrak{n}_+) \simeq \bigoplus_{b+c=a} \text{Pol}^b[\zeta_1, \dots, \zeta_{n-1}] \otimes \text{Pol}^c[\zeta_n],$$

and (4.3) with $N = n-1$, namely,

$$\text{Pol}[\zeta_1, \dots, \zeta_{n-1}] \simeq \bigoplus_{k+2\ell=b} \mathcal{H}^k(\mathbb{C}^{n-1}) \otimes \text{Pol}^\ell[\zeta_1^2 + \cdots + \zeta_{n-1}^2],$$

we have

$$\begin{aligned} & \text{Hom}_{O(n-1)}(V, W \otimes \text{Pol}^a(\mathfrak{n}_+)) \\ & \simeq \bigoplus_{k+2\ell+c=a} \text{Hom}_{O(n-1)}(V, W \otimes \mathcal{H}^k(\mathbb{C}^{n-1})) \otimes \text{Pol}^\ell[\zeta_1^2 + \cdots + \zeta_{n-1}^2] \otimes \text{Pol}^c[\zeta_n]. \end{aligned}$$

Then the statement follows from the bijection (4.6). \square

By the F-method (Proposition 4.1) combined with results on finite-dimensional representations, we obtain a necessary condition for the existence of nonzero differential symmetry breaking operators in the general setting:

Corollary 4.3. *Suppose $(\sigma, V) \in \widehat{M}$, $(\tau, W) \in \widehat{M}'$ and $\lambda, \nu \in \mathbb{C}$. Suppose $\sigma|_{O(1)}$ is a multiple of $\alpha \in \mathbb{Z}/2\mathbb{Z} \simeq \widehat{O(1)}$, and $\tau|_{O(1)}$ is a multiple of $\beta \in \mathbb{Z}/2\mathbb{Z}$, where $O(1)$ denotes the second factor of $M \simeq O(n) \times O(1)$ (or $M' \simeq O(n-1) \times O(1)$). Then*

$$\text{Diff}_{G'} \left(\text{Ind}_P^G(\sigma_\lambda), \text{Ind}_{P'}^{G'}(\tau_\nu) \right) \neq \{0\}$$

only if the following three conditions hold:

$$\nu - \lambda \in \mathbb{N},$$

$$\beta - \alpha \equiv \nu - \lambda \pmod{2},$$

$$\text{Hom}_{O(n-1)}(V, W \otimes \mathcal{H}^k(\mathbb{C}^{n-1})) \neq \{0\} \quad \text{for some } 0 \leq k \leq \nu - \lambda.$$

In particular, if $(\sigma, \tau) \in \widehat{M} \times \widehat{M}'$ satisfies

$$\mathrm{Hom}_{O(n-1)}(V, W \otimes \mathcal{H}(\mathbb{C}^{n-1})) = \{0\},$$

then $\mathrm{Diff}_{G'}(\mathrm{Ind}_P^G(\sigma_\lambda), \mathrm{Ind}_{P'}^{G'}(\tau_\nu)) = \{0\}$ for all $\lambda, \nu \in \mathbb{C}$.

Proof. It follows from Proposition 4.1 that $\mathrm{Diff}_{G'}(\mathrm{Ind}_P^G(\sigma_\lambda), \mathrm{Ind}_{P'}^{G'}(\tau_\nu)) \neq \{0\}$ only if

$$\mathrm{Hom}_{L'}(V, W \otimes \mathrm{Pol}[\zeta_1, \dots, \zeta_n]) \neq \{0\}.$$

First, we consider the action of the first factor $O(n-1)$ of $L' \simeq O(n-1) \times O(1) \times \mathbb{R}$. Then we find $a \in \mathbb{N}$ such that $\mathrm{Hom}_{O(n-1)}(V, W \otimes \mathrm{Pol}^a[\zeta_1, \dots, \zeta_n]) \neq \{0\}$, and therefore $\mathrm{Hom}_{O(n-1)}(V, W \otimes \mathcal{H}^k(\mathbb{C}^{n-1})) \neq \{0\}$ for some k ($0 \leq k \leq a$).

Second, we consider the actions of the second and third factors of L' . Since $e^{tH_0} \in A$ and $-1 \in O(1)$ act on $\mathfrak{n}_+ \simeq \mathbb{C}^n$ as the scalars e^t and -1 , respectively,

$$\mathrm{Hom}_{O(1) \times A}(V, W \otimes \mathrm{Pol}^a[\zeta_1, \dots, \zeta_n]) \neq \{0\}$$

if and only if

$$\nu = \lambda + a \quad \text{and} \quad \beta \equiv \alpha + a \pmod{2}.$$

Thus the corollary is proved. \square

In Chapter 5, we shall prove a necessary and sufficient condition that the space $\mathrm{Hom}_{O(n-1)}(V, W \otimes \mathcal{H}^k(\mathbb{C}^{n-1}))$ does not vanish when $V = \bigwedge^i(\mathbb{C}^n)$ and $W = \bigwedge^j(\mathbb{C}^{n-1})$, and find their explicit generators, see Proposition 5.14.

4.4. Decomposition of the equation $(\widehat{d\pi_{(\sigma, \lambda)^*}}(N_1^+) \otimes \mathrm{id}_W)\psi = 0$.

In Lemma 4.2, we have given a description of $\mathrm{Hom}_{L'}(V, W \otimes \mathrm{Pol}^a(\mathfrak{n}_+))$ by using spherical harmonics. The next step of the matrix-valued F-method in our setting is to write down explicitly the F-system (4.2) according to the canonical decomposition (3.8)

$$\widehat{d\pi_{(\sigma, \lambda)^*}} \otimes \mathrm{id}_W = \widehat{d\pi_{\lambda^*}} \otimes \mathrm{id}_{\mathrm{Hom}(V, W)} + A_\sigma \otimes \mathrm{id}_W.$$

The main result (Proposition 4.4) of this section asserts that the differential operator whose symbol is in (4.2) is given by

Gegenbauer-type operators + matrix-valued vector fields.

To be precise, we introduce the following differential operator of second order

$$(4.7) \quad R_\ell^\lambda := -\frac{1}{2} \left((1+t^2) \frac{d^2}{dt^2} + (1+2\lambda)t \frac{d}{dt} - \ell(\ell+2\lambda) \right)$$

with parameters $\lambda \in \mathbb{C}$ and $\ell \in \mathbb{N}$. Note that $R_\ell^\lambda g(t) = 0$ is the “imaginary” Gegenbauer differential equation (see Lemma 14.3).

Proposition 4.4. *Let $G = O(n+1, 1)$, $(\sigma, V) \in \widehat{M}$, $\lambda \in \mathbb{C}$, and W be a vector space over \mathbb{C} . Suppose $0 \leq k \leq a$ and $\psi = (T_{a-k}g_k)H^{(k)}$ with $g_k(t) \in \text{Pol}_{a-k}[t]_{\text{even}}$ and $H^{(k)} \in \text{Hom}_{\mathbb{C}}(V, W \otimes \mathcal{H}^k(\mathbb{C}^{n-1}))$. Then,*

$$(1) \quad (\widehat{d\pi_{\lambda^*}}(N_1^+) \otimes \text{id}_W)\psi = \frac{\zeta_1}{Q_{n-1}(\zeta')} T_{a-k} \left(R_{a-k}^{\lambda - \frac{n-1}{2}} g_k \right) H^{(k)} + (\lambda + a - 1)(T_{a-k}g_k) \frac{\partial H^{(k)}}{\partial \zeta_1},$$

$$(2) \quad (A_{\sigma}(N_1^+) \otimes \text{id}_W)\psi = \sum_{\ell=1}^n \frac{\partial}{\partial \zeta_{\ell}} (T_{a-k}g_k) H^{(k)} \circ d\sigma(X_{\ell 1}).$$

The rest of this section is devoted to the proof of Proposition 4.4. We note that the L' -intertwining property of the linear maps $H^{(k)}$ is not used in Proposition 4.4.

We begin with an explicit formula of the canonical decomposition (3.8) of $\widehat{d\pi_{(\sigma, \lambda)^*}}$. We define the Euler homogeneity operator on \mathbb{C}^n by

$$E_{\zeta} := \sum_{\ell=1}^n \zeta_{\ell} \frac{\partial}{\partial \zeta_{\ell}}.$$

Then we have:

Lemma 4.5. *Let $G = O(n+1, 1)$ and $\{N_1^+, \dots, N_n^+\}$ be the basis of $\mathfrak{n}_+(\mathbb{R})$, see (2.2). Suppose $(\sigma, V) \in \widehat{M}$ and $\lambda \in \mathbb{C}$. Then the decomposition (3.8) amounts to*

$$\widehat{d\pi_{(\sigma, \lambda)^*}}(N_m^+) = \widehat{d\pi_{\lambda^*}}(N_m^+) \otimes \text{id}_{V^{\vee}} + A_{\sigma}(N_m^+) \quad (1 \leq m \leq n),$$

where

$$(4.8) \quad \widehat{d\pi_{\lambda^*}}(N_m^+) = \lambda \frac{\partial}{\partial \zeta_m} + E_{\zeta} \frac{\partial}{\partial \zeta_m} - \frac{1}{2} \zeta_m \Delta_{\mathbb{C}^n},$$

$$(4.9) \quad A_{\sigma}(N_m^+)F = \sum_{\ell=1}^n \frac{\partial}{\partial \zeta_{\ell}} F \circ d\sigma(X_{\ell m}) \quad \text{for } F \in \text{Pol}(\mathfrak{n}_+) \otimes V^{\vee}.$$

Proof. The “scalar part” is given in [21, Lem. 6.5].

According to Proposition 3.5 and (2.3), the vector part $A_{\sigma}(N_m^+)$ is given by

$$\begin{aligned} A_{\sigma}(N_m^+)F &= - \sum_{\ell=1}^n \frac{\partial}{\partial \zeta_{\ell}} F \circ d\sigma([N_m^+, N_{\ell}^-]_{\mathfrak{m}}) \\ &= - \sum_{\ell=1}^n \frac{\partial}{\partial \zeta_{\ell}} F \circ d\sigma(X_{m\ell}) \\ &= \sum_{\ell=1}^n \frac{\partial}{\partial \zeta_{\ell}} F \circ d\sigma(X_{\ell m}). \end{aligned}$$

□

Thus, the second assertion on the vector part of Proposition 4.4 is proved. In order to show the first assertion on the scalar part, we give a useful formula for the action of the second-order differential operator $\widehat{d\pi_{\lambda^*}}(N_m^+)$ on $\text{Pol}[\zeta_1, \dots, \zeta_n]$.

Lemma 4.6. *If $f \in \text{Pol}^{a-k}(\mathbb{C}^n)^{O(n-1, \mathbb{C})}$ and $h \in \mathcal{H}^k(\mathbb{C}^{n-1})$, then*

$$\widehat{d\pi_{\lambda^*}}(N_m^+)(fh) = \widehat{d\pi_{\lambda^*}}(N_m^+)(f)h + (\lambda + a - 1)f \frac{\partial h}{\partial \zeta_m} \quad \text{for } 1 \leq m \leq n-1.$$

Proof. By (4.8), we have

$$(4.10) \quad \widehat{d\pi_{\lambda^*}}(N_m^+)(fh) = \lambda \frac{\partial}{\partial \zeta_m}(fh) + E_\zeta \frac{\partial}{\partial \zeta_m}(fh) - \frac{1}{2} \zeta_m \Delta_{\mathbb{C}^n}(fh).$$

Observe that $f \frac{\partial h}{\partial \zeta_m}$ is homogeneous of degree $a-1$, and therefore $E_\zeta \left(f \frac{\partial h}{\partial \zeta_m} \right) = (a-1)f \frac{\partial h}{\partial \zeta_m}$. We also observe that $\Delta_{\mathbb{C}^n}(fh) = (\Delta_{\mathbb{C}^n} f)h + 2 \sum_{\ell=1}^n \frac{\partial f}{\partial \zeta_\ell} \frac{\partial h}{\partial \zeta_\ell}$ because $\Delta_{\mathbb{C}^n} h = 0$. It then follows from a direct computation that (4.10) may be simplified to

$$(4.11) \quad \widehat{d\pi_{\lambda^*}}(N_m^+)(fh) = \widehat{d\pi_{\lambda^*}}(N_m^+)(f)h + (\lambda + a - 1)f \frac{\partial h}{\partial \zeta_m} + \frac{\partial f}{\partial \zeta_m} E_\zeta(h) - \sum_{r=1}^n \zeta_m \frac{\partial f}{\partial \zeta_r} \frac{\partial h}{\partial \zeta_r}.$$

Since the polynomial f is $O(n-1, \mathbb{C})$ -invariant, it is annihilated by the generators X_{mr} of the Lie algebra $\mathfrak{o}(n-1)$ (see (2.1)), that is, $\zeta_m \frac{\partial f}{\partial \zeta_r} = \zeta_r \frac{\partial f}{\partial \zeta_m}$ for all $1 \leq r, m \leq n-1$. Therefore,

$$(4.12) \quad \sum_{r=1}^n \zeta_m \frac{\partial f}{\partial \zeta_r} \frac{\partial h}{\partial \zeta_r} = \sum_{r=1}^n \zeta_r \frac{\partial f}{\partial \zeta_m} \frac{\partial h}{\partial \zeta_r} = \frac{\partial f}{\partial \zeta_m} E_\zeta(h).$$

Now the proposed equality follows from (4.11) and (4.12). \square

Finally, we recall the following formula from [21, Lem. 6.11]:

Lemma 4.7. *Suppose $\ell \in \mathbb{N}$ and $\lambda \in \mathbb{C}$. For any $g \in \text{Pol}_\ell[t]_{\text{even}}$,*

$$\widehat{d\pi_{\lambda^*}}(N_m^+)(T_\ell g) = \frac{\zeta_m}{Q_{n-1}(\zeta')} T_\ell \left(R_\ell^{\lambda - \frac{n-1}{2}} g \right) \quad \text{for } 1 \leq m \leq n-1.$$

We are ready to complete the proof of Proposition 4.4.

Proof of Proposition 4.4. The first statement of Proposition 4.4 follows from (4.8) and Lemmas 4.6 and 4.7. The second statement has been proved in Lemma 4.5. Hence the proof of Proposition 4.4 is completed. \square

4.5. Matrix components in the F-method. For actual computations in later chapters, it is convenient to rewrite Proposition 4.4 by means of matrix coefficients.

Let V be a vector space with a basis $\{e_I\}_{I \in \mathcal{I}}$, W with a basis $\{w_J\}_{J \in \mathcal{J}}$, and $\{w_J^\vee\}_{J \in \mathcal{J}}$ denote the dual basis in W^\vee . Given a linear map $T: V \rightarrow W$ we define its matrix coefficient by

$$T_{IJ} := \langle T(e_I), w_J^\vee \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical pairing between W and W^\vee . Clearly, we have for $S \in W^\vee$

$$(4.13) \quad (S \circ T)(e_I) = \sum_{J \in \mathcal{J}} S(w_J) T_{IJ}.$$

Suppose that (σ, V) is a finite-dimensional representation of $M (\simeq O(n) \times O(1))$. We introduce a holomorphic vector field on \mathfrak{n}_+ by

$$(4.14) \quad A_{II'} \equiv A_{II'}^\sigma := \sum_{\ell=1}^n (d\sigma(X_{\ell 1})_{II'}) \frac{\partial}{\partial \zeta_\ell},$$

with respect to the basis $\{e_I\}_{I \in \mathcal{I}}$ of V and the dual basis $\{e_{I'}^\vee\}_{I' \in \mathcal{I}}$ of V^\vee . Then $\{A_{II'}\}$ is the matrix expression of the vector part $A_\sigma(N_1^+)$ of $\widehat{d\pi_{(\sigma, \lambda)}^*}(N_1^+)$ in the following sense.

Lemma 4.8. *Recall that $A_\sigma: \mathfrak{g} \rightarrow \mathcal{D}(\mathfrak{n}_+) \otimes \text{End}(V^\vee)$ is defined by (3.8). Suppose $F(\zeta) = \sum_I F_I(\zeta) e_I^\vee \in \text{Pol}(\mathfrak{n}_+) \otimes V^\vee$. Then $A_\sigma(N_1^+)F$ is given by*

$$A_\sigma(N_1^+)F = \sum_{I \in \mathcal{I}} \left(\sum_{I' \in \mathcal{I}} A_{II'} F_{I'} \right) e_I^\vee.$$

Proof. By (4.13), $(e_{I'}^\vee \circ d\sigma(X_{\ell m}))(e_I) = d\sigma(X_{\ell m})_{II'}$. By (3.9) and (2.3), we have

$$(A_\sigma(N_1^+)F)(e_I) = - \sum_{I' \in \mathcal{I}} \sum_{\ell} \frac{\partial}{\partial \zeta_\ell} F_{I'}(\zeta) d\sigma(X_{1\ell})_{II'} = \sum_{I' \in \mathcal{I}} A_{II'} F_{I'}.$$

□

Given $\psi \in \text{Hom}_{\mathbb{C}}(V, W \otimes \text{Pol}(\mathfrak{n}_+))$, we write

$$\begin{aligned} \psi &= \sum_{I, J} \psi_{IJ} e_I^\vee \otimes w_J, \\ \widehat{d\pi_{(\sigma, \lambda)}^*}(N_1^+) \psi &= \sum_{I, J} M_{IJ} e_I^\vee \otimes w_J, \end{aligned}$$

for some polynomials $\psi_{IJ}(\zeta), M_{IJ}(\zeta) \in \text{Pol}(\mathfrak{n}_+)$. Then the (I, J) -components M_{IJ} of $\widehat{d\pi_{(\sigma, \lambda)}^*}(N_1^+) \psi$ can be computed from $\{\psi_{IJ}\}$ by the following formula.

Proposition 4.9. *Let ψ_{IJ} and M_{IJ} be the matrix coefficients of ψ and of $\widehat{d\pi_{(\sigma, \lambda^*)}(N_1^+) \psi}$ with respect to the basis $\{e_I\}$ of V and $\{w_J\}$ of W . Then we have $M_{IJ} = M_{IJ}^{\text{scalar}} + M_{IJ}^{\text{vect}}$ if we set*

$$\begin{aligned} M_{IJ}^{\text{scalar}} &= \left(\lambda \frac{\partial}{\partial \zeta_1} + E_\zeta \frac{\partial}{\partial \zeta_1} - \frac{1}{2} \zeta_1 \Delta_{\mathbb{C}^n} \right) \psi_{IJ}, \\ M_{IJ}^{\text{vect}} &= \sum_{I'} A_{II'} \psi_{I'J}, \end{aligned}$$

where $A_{II'}$ is a vector field defined in (4.14). In particular, if ψ is of the form

$$\psi = (T_{a-k} g_k) H^{(k)}$$

with $g_k(t) \in \text{Pol}_{a-k}[t]_{\text{even}}$ and $H^{(k)} = \sum_{I,J} H_{IJ}^{(k)} e_I^\vee \otimes w_J \in \text{Hom}(V, W \otimes \mathcal{H}^k(\mathbb{C}^{n-1}))$ for some $0 \leq k \leq a$, then

$$\begin{aligned} M_{IJ}^{\text{scalar}} &= \frac{\zeta_1}{Q_{n-1}(\zeta')} T_{a-k} \left(R_{a-k}^{\lambda - \frac{n-1}{2}} g_k \right) H_{IJ}^{(k)} + (\lambda + a - 1) (T_{a-k} g_k) \frac{\partial H_{IJ}^{(k)}}{\partial \zeta_1}, \\ M_{IJ}^{\text{vect}} &= \sum_{I'} A_{II'} (T_{a-k} g_k) H_{I'J}^{(k)} \\ &= \sum_{I'} \sum_{\ell=1}^n d\sigma(X_{\ell 1})_{II'} \frac{\partial}{\partial \zeta_\ell} \left((T_{a-k} g_k) H_{I'J}^{(k)} \right). \end{aligned}$$

Proof. Immediate from Lemmas 4.5 and 4.8. □

In Chapters 6 and 7, we shall address the matrix-valued differential equation (4.2) in Proposition 4.1 for $V = \bigwedge^i(\mathbb{C}^n)$ and $W = \bigwedge^j(\mathbb{C}^{n-1})$ by solving the system of ordinary differential equations for $\{\psi_{IJ}\}$

$$M_{IJ}^{\text{scalar}} + M_{IJ}^{\text{vect}} = 0$$

for all the indices I and J of the bases of V and W , respectively.

5. APPLICATION OF FINITE-DIMENSIONAL REPRESENTATION THEORY

In this chapter we prepare some results on finite-dimensional representations that will be used in applying the general theory developed in Chapters 3 and 4 to symmetry breaking operators for differential forms.

For this, we construct an explicit basis of $\text{Hom}_{O(n-1)}(V, W \otimes \text{Pol}[\zeta_1, \dots, \zeta_n])$ for $V = \bigwedge^i(\mathbb{C}^n)$ and $W = \bigwedge^j(\mathbb{C}^{n-1})$ (see Proposition 5.17). The key ingredient of the proof is to determine $O(N)$ -invariant elements in the triple tensor product $\bigwedge^i(\mathbb{C}^N) \otimes \bigwedge^j(\mathbb{C}^N) \otimes \mathcal{H}^k(\mathbb{C}^N)$, which is carried out in Section 5.4, see Lemma 5.6 and Proposition 5.7.

At the end of this chapter, we give a proof of the (easy) implication (i) \Rightarrow (iii) in Theorem 2.8.

5.1. Signatures in index sets.

We fix some set theoretic notation. Given a set S , let $|S|$ denote the cardinality of elements in S . We denote by $S \setminus T$ the relative complement of T in S for given two sets S and T , that is, $S \setminus T := \{x \in S : x \notin T\}$.

For $k \in \{1, \dots, N\}$, we set

$$(5.1) \quad \mathcal{I}_{N,k} := \{R \subset \{1, \dots, N\} : |R| = k\}.$$

It is convenient to define $\mathcal{I}_{N,0}$ as $\mathcal{I}_{N,0} := \{\emptyset\}$.

Definition 5.1. For $I \subset \{1, \dots, N\}$ and $p, q \in \mathbb{N}$, we set

$$\begin{aligned} \text{sgn}(I; p) &:= (-1)^{|\{r \in I : r < p\}|}, \\ \text{sgn}(I; p, q) &:= (-1)^{|\{r \in I : \min(p, q) < r < \max(p, q)\}|}. \end{aligned}$$

Here are some basic formulæ for $\text{sgn}(I; p)$ and $\text{sgn}(I; p, q)$.

Lemma 5.2. For $I \subset \{1, \dots, N\}$ and $p, q \in \mathbb{N}$, we have

- (1) $\text{sgn}(I; p) = \text{sgn}(I \cup \{p\}; p)$;
- (2) $\text{sgn}(I; p, q) = \text{sgn}(I \cup \{p\}; p, q) = \text{sgn}(I \cup \{q\}; p, q)$;
- (3) $\text{sgn}(I; p)\text{sgn}(I; q)\text{sgn}(I; p, q) = \begin{cases} +1 & \text{if } \min(p, q) \notin I, \\ -1 & \text{if } \min(p, q) \in I; \end{cases}$
- (4) $\text{sgn}(I \cup \{p\}; q)\text{sgn}(I; p) + \text{sgn}(I \cup \{q\}; p)\text{sgn}(I; q) = 0$ for $p, q \notin I$.

Proof. The proof is a straightforward computation. □

Note that, by Lemma 5.2 (2) and (3), for $I \in \mathcal{I}_{N,i}$ with $N \in I$, the following identity holds:

$$(5.2) \quad \text{sgn}(I \setminus \{N\}; p) = \begin{cases} (-1)^{i-1} \text{sgn}(I; p, N) & \text{if } p \notin I, \\ (-1)^i \text{sgn}(I; p, N) & \text{if } p \in I. \end{cases}$$

5.2. Action of $O(N)$ on the exterior algebra $\bigwedge^*(\mathbb{C}^N)$. Let $\{e_1, \dots, e_N\}$ be the standard basis of \mathbb{C}^N . Given $I = \{i_1, \dots, i_k\} \subset \{1, \dots, N\}$ with $i_1 < \dots < i_k$, we form the standard basis $\{e_I\}$ of $\bigwedge^k(\mathbb{C}^N)$ by setting

$$e_I := e_{i_1} \wedge \dots \wedge e_{i_k}.$$

The natural action σ of $O(N)$ on \mathbb{C}^N induces the exterior representation on $\bigwedge^i(\mathbb{C}^N)$, to be denoted by the same letter σ . Let $X_{pq} = -E_{pq} + E_{qp} \in \mathfrak{o}(N)$ ($1 \leq p \neq q \leq N$) as in (2.1). The matrix coefficient $d\sigma(X_{pq})_{II'} = \langle d\sigma(X_{pq})(e_I), e_{I'}^\vee \rangle$ of the infinitesimal representation $d\sigma$ is given by

$$(5.3) \quad d\sigma(X_{pq})_{II'} = \begin{cases} \operatorname{sgn}(I; p, q) & \text{if } I = J \cup \{p\}, I' = J \cup \{q\}, \\ -\operatorname{sgn}(I; p, q) & \text{if } I = J \cup \{q\}, I' = J \cup \{p\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $J := I \cap I'$ in the first two cases.

Recall from Section 3.3 that, given a representation (σ, V) of $M \simeq O(n) \times O(1)$ and $\lambda \in \mathfrak{a}^*$, we denote by $\widehat{d\pi_{(\sigma, \lambda)^*}}$ the algebraic Fourier transform of the Lie algebra homomorphism $d\pi_{(\sigma, \lambda)^*} : \mathfrak{g} \rightarrow \mathcal{D}(\mathfrak{n}_-) \otimes \operatorname{End}(V^\vee)$. When $\sigma = \bigwedge^i(\mathbb{C}^n)$ and $\sigma|_{O(n)}$ is the exterior representation, we write simply $\widehat{d\pi_{(i, \lambda)^*}}$ for $\widehat{d\pi_{(\sigma, \lambda)^*}}$, as it is independent of the restriction of σ to the second factor $O(1)$. Then the matrix components $A_{II'}$ of the vector part of $\widehat{d\pi_{(i, \lambda)^*}}(N_1^+)$ (see Lemma 4.8) takes the following form:

Lemma 5.3. *Let $I, I' \in \mathcal{I}_{n, i}$. Then the (I, I') -component $A_{II'}$ of the vector part of $\widehat{d\pi_{(i, \lambda)^*}}(N_1^+)$ is given by the following vector field*

$$A_{II'} = \begin{cases} \operatorname{sgn}(I; \ell) \frac{\partial}{\partial x_\ell} & \text{if } (I \setminus I') \amalg (I' \setminus I) = \{1, \ell\} \ (\ell \neq 1), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We recall from (4.14) that $A_{II'} = \sum_{\ell=1}^n d\sigma(X_{\ell 1})_{II'} \frac{\partial}{\partial x_\ell}$. Hence the lemma is clear from (5.3). \square

Example 5.4. *With respect to the basis $\{dx_2 \wedge dx_3, dx_1 \wedge dx_3, dx_1 \wedge dx_2\}$ of $\mathcal{E}^2(\mathbb{R}^3)$ as a $C^\infty(\mathbb{R}^3)$ -module, the vector part of $\widehat{d\pi_{(i, \lambda)^*}}(N_1^+)$ with $i = 2$ acts on $F = \sum_{1 \leq k < \ell \leq 3} F_{k\ell} dx_k \wedge dx_\ell \in \mathcal{E}^2(\mathbb{R}^3) \simeq C^\infty(\mathbb{R}^3) \otimes \mathbb{C}^3$ by*

$$\begin{pmatrix} F_{23} \\ F_{13} \\ F_{12} \end{pmatrix} \mapsto \begin{pmatrix} 0 & \frac{\partial}{\partial \zeta_2} & -\frac{\partial}{\partial \zeta_3} \\ -\frac{\partial}{\partial \zeta_2} & 0 & 0 \\ \frac{\partial}{\partial \zeta_3} & 0 & 0 \end{pmatrix} \begin{pmatrix} F_{23} \\ F_{13} \\ F_{12} \end{pmatrix}.$$

5.3. Construction of intertwining operators. For $V = \bigwedge^i(\mathbb{C}^N)$ and $W = \bigwedge^j(\mathbb{C}^N)$, we shall construct building blocks of $O(N)$ -equivariant bilinear maps

$$B^{(k)}: \bigwedge^i(\mathbb{C}^N) \times \bigwedge^j(\mathbb{C}^N) \longrightarrow \text{Pol}[\zeta_1, \dots, \zeta_N],$$

as follows. Suppose $j = i$. For $I, I' \in \mathcal{I}_{N,i}$, we define \mathbb{C} -bilinear maps $B^{(0)}$ and $B^{(2)}$ by giving the images of the basis elements:

$$(5.4) \quad B^{(0)}(e_I, e_{I'}) := \begin{cases} 1 & \text{if } I = I', \\ 0 & \text{otherwise.} \end{cases}$$

$$(5.5) \quad B^{(2)}(e_I, e_{I'}) := \begin{cases} \sum_{\ell \in I} \zeta_\ell^2 & \text{if } I = I', \\ \text{sgn}(J; p, q) \zeta_p \zeta_q & \text{if } I = J \cup \{p\}, I' = J \cup \{q\}, p \neq q, \\ 0 & \text{if } |(I \setminus I')| > 1. \end{cases}$$

Suppose $j = i - 1$. For $I \in \mathcal{I}_{N,i}$ and $J \in \mathcal{I}_{N,i-1}$, we set

$$(5.6) \quad B^{(1)}(e_I, e_J) := \begin{cases} \text{sgn}(J; \ell) \zeta_\ell & \text{if } I = J \cup \{\ell\}, \\ 0 & \text{if } J \not\subset I. \end{cases}$$

Lemma 5.5. *The bilinear maps $B^{(k)}$ ($k = 0, 1, 2$) are $O(N)$ -equivariant, namely,*

$$B^{(k)}(gv, gw)(g\zeta) = B^{(k)}(v, w)(\zeta)$$

for all $g \in O(N)$, $v \in V$, $w \in W$, and $\zeta = (\zeta_1, \dots, \zeta_N)$.

We could prove Lemma 5.5 directly by the formula (5.3), but we shall give an alternative and simpler proof in Section 8.6 by using the symbol map for $O(N)$ -equivariant differential operators.

Since $\bigwedge^i(\mathbb{C}^N)$ is self-dual as an $O(N)$ -module (cf. (8.4)), the bilinear forms $B^{(k)}$ induce the following $O(N)$ -equivariant linear maps

$$H_{i \rightarrow j}^{(k)}: \bigwedge^i(\mathbb{C}^N) \rightarrow \bigwedge^j(\mathbb{C}^N) \otimes \text{Pol}^k[\zeta_1, \dots, \zeta_N]$$

given by

$$(5.7) \quad H_{i \rightarrow i}^{(0)}(e_I) := \sum_{I' \in \mathcal{I}_{N,i}} B^{(0)}(e_I, e_{I'}) e_{I'} = e_I,$$

$$(5.8) \quad H_{i \rightarrow i-1}^{(1)}(e_I) := \sum_{J \in \mathcal{I}_{N,i-1}} B^{(1)}(e_I, e_J) e_J = \sum_{\ell \in I} \text{sgn}(I; \ell) e_{I \setminus \{\ell\}} \zeta_\ell,$$

$$(5.9) \quad H_{i-1 \rightarrow i}^{(1)}(e_J) := \sum_{I \in \mathcal{I}_{N,i}} B^{(1)}(e_I, e_J) e_I = \sum_{\ell \notin J} \text{sgn}(J; \ell) e_{J \cup \{\ell\}} \zeta_\ell,$$

$$(5.10) \quad \begin{aligned} H_{i \rightarrow i}^{(2)}(e_I) &:= \sum_{I' \in \mathcal{I}_{N,i}} B^{(2)}(e_I, e_{I'}) e_{I'} \\ &= \left(\sum_{\ell \in I} \zeta_\ell^2 \right) e_I + \sum_{q \notin I} \sum_{p \in I} \text{sgn}(I; p, q) e_{I \setminus \{p\} \cup \{q\}} \zeta_p \zeta_q. \end{aligned}$$

Then all the matrix coefficients of $H_{i \rightarrow j}^{(k)}$ are harmonic polynomials for the first three cases, but not for $H_{i \rightarrow i}^{(2)}$. In order to make the matrix coefficients to be harmonic polynomials, we set

$$(5.11) \quad \begin{aligned} \tilde{H}_{i \rightarrow j}^{(k)} &:= H_{i \rightarrow j}^{(k)} \quad \text{if } j - i = k = 0 \text{ or } |j - i| = k = 1, \\ \tilde{H}_{i \rightarrow i}^{(2)} &:= H_{i \rightarrow i}^{(2)} - \frac{i}{N} Q_N(\zeta) H_{i \rightarrow i}^{(0)}. \end{aligned}$$

Then the matrix coefficients $\left(\tilde{H}_{i \rightarrow i}^{(2)} \right)_{II'} := \left\langle \tilde{H}_{i \rightarrow i}^{(2)}(e_I), e_{I'}^\vee \right\rangle$ ($I, I' \in \mathcal{I}_{N,i}$) are given by

$$\left(\tilde{H}_{i \rightarrow i}^{(2)} \right)_{II'} = \begin{cases} \tilde{Q}_I(\zeta) & \text{if } I = I', \\ \text{sgn}(J; p, q) \zeta_p \zeta_q & \text{if } I = J \cup \{p\}, I' = J \cup \{q\} \text{ with } p \neq q, \\ 0 & \text{otherwise,} \end{cases}$$

where we set

$$(5.12) \quad \tilde{Q}_I(\zeta) := \sum_{\ell \in I} \zeta_\ell^2 - \frac{i}{N} Q_N(\zeta) \quad \text{for } I \in \mathcal{I}_{N,i}.$$

Thus $\left(\tilde{H}_{i \rightarrow i}^{(2)} \right)_{II'}$ are harmonic polynomials for all $I, I' \in \mathcal{I}_{N,i}$. Hence we have defined the linear maps

$$(5.13) \quad \tilde{H}_{i \rightarrow j}^{(k)} : \wedge^i(\mathbb{C}^N) \longrightarrow \wedge^j(\mathbb{C}^N) \otimes \mathcal{H}^k(\mathbb{C}^N),$$

which are obviously $O(N)$ -equivariant in all the cases. In the next section, we shall prove that $\tilde{H}_{i \rightarrow j}^{(k)}$ exhaust all such $O(N)$ -linear maps up to scalars, see Proposition 5.7 below.

We need to be careful at the extremal places where the modified maps $\tilde{H}_{i \rightarrow j}^{(k)}$ may vanish:

$$(5.14) \quad \tilde{H}_{0 \rightarrow 0}^{(2)} = \tilde{H}_{N \rightarrow N}^{(2)} = 0.$$

5.4. Application of finite-dimensional representation theory. In this section we prove that the linear maps $\tilde{H}_{i \rightarrow j}^{(k)}$ introduced in (5.13) exhaust all nonzero $O(N)$ -homomorphisms $\bigwedge^i(\mathbb{C}^N) \rightarrow \bigwedge^j(\mathbb{C}^N) \otimes \mathcal{H}^k(\mathbb{C}^N)$ up to scalar multiplication. The results will be used for actual calculations in solving the F-system, which yield all differential symmetry breaking operators $\mathcal{E}^i(S^n)_{u,\delta} \rightarrow \mathcal{E}^j(S^{n-1})_{v,\varepsilon}$, see Theorems 1.5-1.8. To be more precise, we prove the following.

Lemma 5.6. *Let $N \geq 1$. Then the following three conditions on (i, j, k) with $0 \leq i, j \leq N$ and $k \in \mathbb{N}$ are equivalent.*

- (i) $\text{Hom}_{O(N)}(\bigwedge^i(\mathbb{C}^N), \bigwedge^j(\mathbb{C}^N) \otimes \mathcal{H}^k(\mathbb{C}^N)) \neq \{0\},$
- (ii) $\dim_{\mathbb{C}}(\text{Hom}_{O(N)}(\bigwedge^i(\mathbb{C}^N), \bigwedge^j(\mathbb{C}^N) \otimes \mathcal{H}^k(\mathbb{C}^N))) = 1,$
- (iii) The triple (i, j, k) belongs to one of the following three cases :
 - (a) $i = j$ and $k = 0$.
 - (b) $i = j \in \{1, 2, \dots, N-1\}$ and $k = 2$.
 - (c) $|i - j| = k = 1$.

We observe that nonzero $O(N)$ -homomorphisms $\tilde{H}_{i \rightarrow j}^{(k)}$ were constructed in Section 5.3 for all the triples (i, j, k) appearing in (iii) of Lemma 5.6. Then the multiplicity-free property ((ii) of Lemma 5.6) implies the following proposition.

Proposition 5.7. *Suppose (i, j, k) satisfies one of (therefore all of) the equivalent conditions in Lemma 5.6. Then, we have*

$$\text{Hom}_{O(N)}(\bigwedge^i(\mathbb{C}^N), \bigwedge^j(\mathbb{C}^N) \otimes \mathcal{H}^k(\mathbb{C}^N)) = \mathbb{C} \tilde{H}_{i \rightarrow j}^{(k)}.$$

Remark 5.8. Since $\text{Pol}(\mathbb{C}^N) \simeq \mathbb{C}[Q_N] \otimes \mathcal{H}(\mathbb{C}^N)$ as an $O(N)$ -module (see (4.3)), any $O(N)$ -homomorphism from $\bigwedge^i(\mathbb{C}^N)$ to $\bigwedge^j(\mathbb{C}^N) \otimes \text{Pol}(\mathbb{C}^N)$ can be written as a linear combination of $Q_N^\ell \tilde{H}_{i \rightarrow j}^{(k)}$ ($\ell \in \mathbb{N}$, $k \in \{0, 1, 2\}$).

The rest of this section is devoted to the proof of Lemma 5.6. For this, we observe that $\bigwedge^i(\mathbb{C}^N)$ may be thought of as a $U(N)$ -module, whereas $\mathcal{H}^k(\mathbb{C}^N)$ is just an $O(N)$ -module. Then our strategy is to use the branching laws with respect to a chain of subgroups

$$U(N) \times U(N) \supset U(N) \supset O(N),$$

and the proof is divided into the following two steps.

Step 1. Decompose $\bigwedge^i(\mathbb{C}^N) \otimes \bigwedge^j(\mathbb{C}^N)$ into irreducible $U(N)$ -modules, see Lemma 5.9.

Step 2. Consider the branching law $U(N) \downarrow O(N)$, and find the multiplicities of the $O(N)$ -module $\mathcal{H}^k(\mathbb{C}^N)$ occurring in the irreducible $U(N)$ -summands of the tensor product representation in Step 1, see Lemma 5.10.

We fix some notations. We set

$$\Lambda^+(N) := \{\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{Z}^N : \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0\}.$$

We write $F(U(N), \lambda)$ for the irreducible finite-dimensional representation of $U(N)$ (or equivalently, the irreducible polynomial representation of $GL(N, \mathbb{C})$) with highest weight λ . If λ is of the form $(\underbrace{c_1, \dots, c_1}_{m_1}, \underbrace{c_2, \dots, c_2}_{m_2}, \dots, \underbrace{c_\ell, \dots, c_\ell}_{m_\ell}, 0, \dots, 0)$, then we

also write $\lambda = (c_1^{m_1}, c_2^{m_2}, \dots, c_\ell^{m_\ell})$ as usual. For instance $F(U(N), 1^i) \simeq \bigwedge^i(\mathbb{C}^N)$. As Step 1, we use the following lemma:

Lemma 5.9. *We have the following isomorphisms of $U(N)$ -modules.*

$$\begin{aligned} \bigwedge^i(\mathbb{C}^N) \otimes \bigwedge^i(\mathbb{C}^N) &\simeq \bigoplus_{k=\max(0, 2i-N)}^i F(U(N), (2^k, 1^{2i-2k})), \\ \bigwedge^i(\mathbb{C}^N) \otimes \bigwedge^{i-1}(\mathbb{C}^N) &\simeq \bigoplus_{k=\max(0, 2i-N-1)}^{i-1} F(U(N), (2^k, 1^{2i-2k-1})). \end{aligned}$$

Proof. Both decompositions are given by the skew Pieri rule for the tensor product of the exterior representations $\bigwedge^i(\mathbb{C}^N)$. \square

As Step 2, we consider how each $U(N)$ -irreducible summand in Lemma 5.9 decomposes as an $O(N)$ -module. This decomposition is not always multiplicity-free, however, it turns out that the $O(N)$ -irreducible module $\mathcal{H}^s(\mathbb{C}^N)$ ($s \in \mathbb{N}$) occurs at most once. To be precise, we have the following.

Lemma 5.10. *Let $N \geq 2$. Suppose $s, k, \ell \in \mathbb{N}$ satisfy $k + \ell \leq N$. Then the following three conditions on (s, k, ℓ) are equivalent:*

- (i) $\text{Hom}_{O(N)} \left(\mathcal{H}^s(\mathbb{C}^N), F(U(N), (2^k, 1^\ell)) \Big|_{O(N)} \right) \neq \{0\},$
- (ii) $\dim \text{Hom}_{O(N)} \left(\mathcal{H}^s(\mathbb{C}^N), F(U(N), (2^k, 1^\ell)) \Big|_{O(N)} \right) = 1,$
- (iii) $(s, \ell) = (0, 0)$ with $0 \leq k \leq N$, $(s, \ell) = (1, 1)$ with $0 \leq k \leq N - 1$, or $(s, \ell) = (2, 0)$ with $1 \leq k \leq N$.

For the proof of Lemma 5.10, we need some combinatorics related to representations of $U(N)$ and $O(N)$.

We shall identify $\lambda \in \Lambda^+(N)$ with the corresponding Young diagram. For $\lambda, \nu, \mu \in \Lambda^+(N)$, we denote by $c_{\nu\mu}^\lambda \in \mathbb{N}$ the Littlewood–Richardson coefficient, namely, the structure constant for the product in the \mathbb{C} -algebra of symmetric functions with respect to the basis of Schur functions

$$s_\nu s_\mu = \sum_{\lambda} c_{\nu\mu}^\lambda s_\lambda.$$

We note that $c_{\nu\mu}^\lambda \neq 0$ only if $\nu \subset \lambda$ and $\mu \subset \lambda$, namely, $\nu_j \leq \lambda_j$ and $\mu_j \leq \lambda_j$ for all j ($1 \leq j \leq N$). The Littlewood–Richardson coefficient $c_{\nu\mu}^\lambda$ has a combinatorial description in several ways such as

$$c_{\nu\mu}^\lambda = |\{\text{tableau } T \text{ on skew diagram } \lambda \setminus \nu: \text{weight}(T) = \mu, \text{word}(T) \text{ is a lattice permutation}\}|,$$

where we recall:

- $\lambda \setminus \nu$ is the skew diagram obtained by removing all the boxes of ν from the diagram λ with the same top-left corner;
- a tableau T is a filling of the boxes of a skew diagram with positive integers, weakly increasing in rows and strictly decreasing in columns;
- $\text{weight}(T)$ is a vector such that i -th component equals the times of occurrences of the positive integer i in the tableau T ;
- $\text{word}(T)$ is a sequence of positive integers in T when we read from right to left in successive rows, starting with the top row;
- A sequence a_1, \dots, a_N of positive integers is said to be a lattice permutation if $|\{1 \leq k \leq r : a_k = i\}|$ is a weakly decreasing function of $i \in \mathbb{N}$ for every r ($1 \leq r \leq N$).

We introduce the following map

$$(5.15) \quad \Lambda^+(N) \times \Lambda^+(N) \longrightarrow \mathbb{Z}[\Lambda^+(N)], \quad (\lambda, \nu) \mapsto \lambda/\nu := \bigoplus_{\mu \in \Lambda^+(N)} c_{\nu\mu}^\lambda \mu,$$

where $\mathbb{Z}[S]$ denotes the free \mathbb{Z} -module generated by elements of a set S .

If the skew diagram $\lambda \setminus \nu$ is a Young diagram, namely, if $\nu_j = \lambda_j$ ($1 \leq j \leq k$) and $\nu_j = 0$ ($k+1 \leq j \leq N$) for some k , then it is readily seen from the above combinatorial description that

$$(5.16) \quad c_{\nu\mu}^\lambda = \begin{cases} 1 & \text{if } \mu = \lambda \setminus \nu, \\ 0 & \text{if } \mu \neq \lambda \setminus \nu. \end{cases}$$

Thus, $\lambda/\nu = \lambda \setminus \nu$ if $\lambda \setminus \nu$ is a Young diagram.

We define two subsets of $\Lambda^+(N)$ by

$$\Lambda^+(N)_{\text{even}} := \{\lambda \in \Lambda^+(N) : \lambda_j \in 2\mathbb{Z} \text{ for } 1 \leq j \leq N\},$$

$$\Lambda^+(N)_{\text{BD}} := \{\lambda \in \Lambda^+(N) : \lambda'_1 + \lambda'_2 \leq N\},$$

where $\lambda'_1 := \max\{i : \lambda_i \geq 1\}$ and $\lambda'_2 := \max\{i : \lambda_i \geq 2\}$ for $\lambda = (\lambda_1, \dots, \lambda_N) \in \Lambda^+(N)$. Then λ'_1 is nothing but the maximal column length, denoted also by $\ell(\lambda)$.

It is readily seen that $\Lambda^+(N)_{\text{BD}}$ consists of elements of the following two types:

$$\text{Type I: } (a_1, \dots, a_k, \underbrace{0, \dots, 0}_{N-k}, 0),$$

$$\text{Type II: } (a_1, \dots, a_k, \underbrace{1, \dots, 1}_{N-2k}, \underbrace{0, \dots, 0}_k),$$

with $a_1 \geq a_2 \geq \dots \geq a_k > 0$ and $0 \leq k \leq \lfloor \frac{N}{2} \rfloor$.

Following Weyl ([27, Chap. V, Sect. 7]), we parametrize the set $\widehat{O(N)}$ of equivalence classes of irreducible representations of $O(N)$ as

$$(5.17) \quad \Lambda^+(N)_{\text{BD}} \xrightarrow{\sim} \widehat{O(N)}, \quad \lambda \mapsto [\lambda],$$

where $[\lambda]$ is the $O(N)$ -irreducible summand of $F(U(N), \lambda)$ which contains the highest weight vector.

Example 5.11. $\mathcal{H}^s(\mathbb{C}^N) = [s]$ ($s \in \mathbb{N}$), and $\bigwedge^\ell(\mathbb{C}^N) = [1^\ell]$ ($0 \leq \ell \leq N$).

Moreover, Types I and II are related by the following $O(N)$ -isomorphism:

$$(5.18) \quad [(a_1, \dots, a_k, 1, \dots, 1, 0, \dots, 0)] = \det \otimes [(a_1, \dots, a_k, 0, \dots, 0)].$$

The restriction of the $O(N)$ -module $[(a_1, \dots, a_k, 0, \dots, 0)]$ to the subgroup $SO(N)$ is reducible if and only if $N = 2k$. In this case we have:

$$[(a_1, \dots, a_k, 0, \dots, 0)]|_{SO(N)} = F(SO(N), (a_1, \dots, a_k)) \oplus F(SO(N), (a_1, \dots, a_{k-1}, -a_k)).$$

For $\lambda \notin \Lambda^+(N)_{\text{BD}}$, we apply the $O(N)$ -modification rule which is a map

$$(5.19) \quad \Lambda^+(N) \setminus \Lambda^+(N)_{\text{BD}} \longrightarrow \mathbb{Z}[\widehat{O(N)}], \quad \lambda \mapsto [\lambda]$$

constructed as follows (see [12, Sect. 3], [23]). If $\ell(\lambda) \geq \lfloor \frac{N}{2} \rfloor$ then we define $\tilde{\lambda}$ to be the removal of a continuous boundary hook of length $h := 2\ell(\lambda) - N$ and row length x , starting in the first column of the Young diagram associated to λ . We set $[\lambda] := 0$ if $\tilde{\lambda}$ is not a Young diagram; $[\lambda] := (-1)^x \det \otimes [\tilde{\lambda}]$ if $\tilde{\lambda} \in \Lambda^+(N)_{\text{BD}}$. Otherwise, we repeat this procedure to $\tilde{\lambda} \in \Lambda^+(N) \setminus \Lambda^+(N)_{\text{BD}}$.

We note that $\Lambda^+(N)_{\text{BD}}$ contains elements λ with $\ell(\lambda) \geq \lfloor \frac{N}{2} \rfloor$, namely, elements of Type II. The $O(N)$ -modification rule also applies to these elements, and yields the isomorphism (5.18). In fact, suppose $\lambda \in \Lambda^+(N)_{\text{BD}}$ is of Type II, say $\lambda =$

$(a_1, \dots, a_k, 1, \dots, 1, 0, \dots, 0)$. In this case, $\ell(\lambda) = n - k (\geq \lfloor \frac{n}{2} \rfloor)$, $h = 2(n - k) - n = n - 2k$ and $x = 0$, and thus $\tilde{\lambda} = (a_1, \dots, a_k, 0, \dots, 0)$. Thus the $O(N)$ -modification rule in this special case gives rise to the isomorphism (5.18).

Combining (5.17) and (5.19), we get a \mathbb{Z} -linear map

$$(5.20) \quad \mathbb{Z}[\Lambda^+(N)] \longrightarrow \mathbb{Z}[\widehat{O(N)}], \quad \lambda \mapsto [\lambda].$$

For $\lambda \in \Lambda^+(N)$, the representation $F(U(N), \lambda)$ decomposes as an $O(N)$ -module in accordance with the $O(N)$ -modification rule applied to the universal character formula [12], [23]:

$$(5.21) \quad F(U(N), \lambda)|_{O(N)} \simeq \bigoplus_{\nu \in \Lambda^+(N)_{\text{even}}} [\lambda/\nu],$$

where λ/ν is defined in (5.15) as an element of $\mathbb{Z}[\Lambda^+(N)]$. For the proof of Lemma 5.10, we use the following two claims.

Claim 5.12. Suppose $\lambda = (2^k, 1^\ell) \in \Lambda^+(N)$ and $\nu \in \Lambda^+(N)_{\text{even}}$ with $\nu \subset \lambda$. Then ν is of the form $\nu = (2^{k-r})$ for some $0 \leq r \leq k$ and $\lambda/\nu = (2^r, 1^k)$.

Proof of Claim 5.12. The first assertion is clear because $\nu \subset (2^k, 1^\ell)$ and $\nu \in \Lambda^+(N)_{\text{even}}$. Then, the skew diagram $\lambda \setminus \nu$ is actually a Young diagram $(2^r, 1^k)$, and therefore, the claim follows from (5.16). \square

We write $\text{pr}_{\mathcal{H}}: \mathbb{Z}[\widehat{O(N)}] \longrightarrow \mathbb{Z}[\{\mathcal{H}^s(\mathbb{C}^N) : s \in \mathbb{N}\}]$ for the canonical projection.

Claim 5.13. Suppose $\lambda = (2^r, 1^\ell) \in \Lambda^+(N)$. Then we have

$$\text{pr}_{\mathcal{H}}([\lambda]) = \begin{cases} \mathcal{H}^{2r+\ell}(\mathbb{C}^N) & \text{if } (r, \ell) = (0, 0), (0, 1), \text{ or } (1, 0), \\ 0 & \text{otherwise.} \end{cases}$$

Proof of Claim 5.13. The assertion is obvious from the bijection (5.17) if $\lambda \in \Lambda^+(N)_{\text{BD}}$.

What remains to prove is $\text{pr}_{\mathcal{H}}([\lambda]) = 0$ for $\lambda \notin \Lambda^+(N)_{\text{BD}}$. First of all, we see from the $O(N)$ -modification rule (5.19) that the $\mathcal{H}^s(\mathbb{C}^N)$ -component of $[\lambda]$ is nonzero only if $s \in \{0, 1, 2\}$ corresponding to \emptyset , \square or $\square\square$. Further, $s = |\lambda| - h$ and $h = 2(r + \ell) - N (> 0)$. Hence $\ell = N - s$.

For $s = 0$, we have $\ell = N$, and therefore, the only possible form of λ is $\lambda = (1^N)$. Hence the corresponding $O(N)$ -representation is $[\lambda] = \det \not\simeq \mathbb{1}$.

For $s = 1$, we have $\ell = N - 1$, and therefore, the only possible forms of λ are either (1^{N-1}) with $N \geq 3$ or $(2^1, 1^{N-1})$. Then $[\lambda] \simeq \det \otimes \mathcal{H}^1(\mathbb{C}^N)$ or $\{0\}$, respectively, by the $O(N)$ -modification rule (5.19). Thus $\text{pr}_{\mathcal{H}}([\lambda]) = 0$ in either case.

For $s = 2$, we have $\ell = N - 2$, and therefore, the only possible forms of λ are either $\lambda = (2^1, 1^{N-2})$ with $N \geq 3$ or $(2^2, 1^{N-2})$. Then $[\lambda] = \det \otimes \mathcal{H}^2(\mathbb{C}^N)$ or

$-\det \otimes \mathcal{H}^2(\mathbb{C}^N)$, respectively, again by the $O(N)$ -modification rule (5.19). Hence, we have $\text{pr}_{\mathcal{H}}([\lambda]) = 0$ in either case. Thus the claim is shown. \square

We are ready to complete the proof of Lemma 5.10.

Proof of Lemma 5.10. By the branching law (5.21) for the restriction $U(N) \downarrow O(N)$, we have from Claim 5.12

$$F(U(N), (2^k, 1^\ell))|_{O(N)} \simeq \bigoplus_{r=0}^k [(2^r, 1^\ell)].$$

Therefore

$$\text{pr}_{\mathcal{H}}(F(U(N), (2^k, 1^\ell))|_{O(N)}) = \begin{cases} \mathcal{H}^0(\mathbb{C}^N) & (k, \ell) = (0, 0), \\ \mathcal{H}^1(\mathbb{C}^N) & \ell = 1, k \geq 0, \\ \mathcal{H}^0(\mathbb{C}^N) \oplus \mathcal{H}^2(\mathbb{C}^N) & \ell = 0, k \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the lemma is proved. \square

5.5. Classification of $\text{Hom}_{O(n-1)}(\bigwedge^i(\mathbb{C}^n), \bigwedge^j(\mathbb{C}^{n-1}) \otimes \mathcal{H}^k(\mathbb{C}^{n-1}))$. We recall that the group $O(n-1)$ acts on $\mathfrak{n}_+ \simeq \mathbb{C}^n$ stabilizing the last coordinate ζ_n , and thus acts also on $\mathfrak{n}'_+ = \mathfrak{n}_+ \cap \{\zeta_n = 0\} \simeq \mathbb{C}^{n-1}$, and thus we have an isomorphism $\text{Pol}(\mathfrak{n}_+) \simeq \text{Pol}[\zeta_1, \dots, \zeta_{n-1}] \otimes \text{Pol}[\zeta_n]$ as an $O(n-1)$ -module. In this section we determine explicitly $\text{Hom}_{O(n-1)}(V|_{O(n-1)}, W \otimes \mathcal{H}(\mathbb{C}^{n-1}))$ for the $O(n)$ -module $V = \bigwedge^i(\mathbb{C}^n)$ and the $O(n-1)$ -module $W = \bigwedge^j(\mathbb{C}^{n-1})$. The results will play a basic role in the classification of all differential symmetry breaking operators $\mathcal{E}^i(S^n)_{u,\delta} \longrightarrow \mathcal{E}^j(S^{n-1})_{v,\varepsilon}$.

The main result of this section is the following:

Proposition 5.14. *Let $n \geq 2$. Suppose that $0 \leq i \leq n$, $0 \leq j \leq n-1$, and $k \in \mathbb{N}$. Then the following three conditions on (i, j, k) are equivalent.*

(i) $\text{Hom}_{O(n-1)}(\bigwedge^i(\mathbb{C}^n), \bigwedge^j(\mathbb{C}^{n-1}) \otimes \mathcal{H}^k(\mathbb{C}^{n-1})) \neq \{0\}.$

(ii) $\dim \text{Hom}_{O(n-1)}(\bigwedge^i(\mathbb{C}^n), \bigwedge^j(\mathbb{C}^{n-1}) \otimes \mathcal{H}^k(\mathbb{C}^{n-1})) = 1.$

(iii) *The triple (i, j, k) belongs to one of the following cases:*

Case 1: $j = i - 2$ ($2 \leq i \leq n$), $k = 1$,

Case 2: $j = i - 1$

2-a: $i = 1$, $k = 0, 1$,

2-b: $2 \leq i \leq n-1$, $k = 0, 1, 2$,

2-c: $i = n$, $k = 0$,

Case 3: $j = i$:

- \mathcal{B} -a: $i = 0, k = 0,$
 \mathcal{B} -b: $1 \leq i \leq n - 2, k = 0, 1, 2,$
 \mathcal{B} -c: $i = n - 1, k = 0, 1,$

Case 4: $j = i + 1$ ($0 \leq i \leq n - 2$), $k = 1$.

Explicit generator $h_{i \rightarrow j}^{(k)}$ in $\text{Hom}_{O(n-1)}(\bigwedge^i(\mathbb{C}^n), \bigwedge^j(\mathbb{C}^{n-1}) \otimes \mathcal{H}^k(\mathbb{C}^{n-1}))$ will be given in (5.24)–(5.27) below. We begin with the following elementary lemma:

Lemma 5.15. *As an $O(n-1, \mathbb{C})$ -module, $V = \bigwedge^i(\mathbb{C}^n)$ decomposes as*

$$\bigwedge^i(\mathbb{C}^n) = \bigwedge^i(\mathbb{C}^{n-1}) \oplus \bigwedge^{i-1}(\mathbb{C}^{n-1}).$$

The spaces $\bigwedge^i(\mathbb{C}^{n-1})$ and $\bigwedge^{i-1}(\mathbb{C}^{n-1})$ have bases $\{e_I : I \in \mathcal{I}_{n-1,i}\}$ and $\{e_I : I \in \mathcal{I}_{n-1,i-1}\}$, respectively. We normalize the first and the second projections by

$$(5.22) \quad \text{pr}_{i \rightarrow i}(e_I) := \begin{cases} e_I & \text{if } n \notin I, \\ 0 & \text{if } n \in I, \end{cases}$$

$$(5.23) \quad \text{pr}_{i \rightarrow i-1}(e_I) := \begin{cases} 0 & \text{if } n \notin I, \\ (-1)^{i-1} e_{I \setminus \{n\}} & \text{if } n \in I. \end{cases}$$

The signature of $\text{pr}_{i \rightarrow i-1}$ is taken in a way that it fits with the interior multiplication $\iota_{\frac{\partial}{\partial x_n}}$ for differential forms (see (8.18)).

Proof of Proposition 5.14. By Lemma 5.15, the proof reduces to Lemma 5.6 with $N = n - 1$. In fact, explicit generator $h_{i \rightarrow j}^{(k)}$ is given as follows:

(5.24)

$$\text{Case } j = i - 2: h_{i \rightarrow i-2}^{(1)} := \tilde{H}_{i-1 \rightarrow i-2}^{(1)} \circ \text{pr}_{i \rightarrow i-1}.$$

(5.25)

$$\text{Case } j = i - 1: h_{i \rightarrow i-1}^{(k)} := \tilde{H}_{i-1 \rightarrow i-1}^{(k)} \circ \text{pr}_{i \rightarrow i-1} \quad (k = 0, 2), \quad h_{i \rightarrow i-1}^{(1)} := \tilde{H}_{i \rightarrow i-1}^{(1)} \circ \text{pr}_{i \rightarrow i}.$$

(5.26)

$$\text{Case } j = i: h_{i \rightarrow i}^{(k)} := \tilde{H}_{i \rightarrow i}^{(k)} \circ \text{pr}_{i \rightarrow i} \quad (k = 0, 2), \quad h_{i \rightarrow i}^{(1)} := \tilde{H}_{i-1 \rightarrow i}^{(1)} \circ \text{pr}_{i \rightarrow i-1}.$$

(5.27)

$$\text{Case } j = i + 1: h_{i \rightarrow i+1}^{(1)} := \tilde{H}_{i \rightarrow i+1}^{(1)} \circ \text{pr}_{i \rightarrow i}.$$

Here we have applied (5.13) to $N = n - 1$ for $\tilde{H}_{i \rightarrow j}^{(k)}$ in the above formula.

We see from (5.14) that some of these operators vanish, namely,

$$(5.28) \quad h_{1 \rightarrow 0}^{(2)} = 0, \quad h_{n \rightarrow n-1}^{(1)} = h_{n \rightarrow n-1}^{(2)} = 0, \quad h_{0 \rightarrow 0}^{(1)} = h_{0 \rightarrow 0}^{(2)} = 0, \quad h_{n-1 \rightarrow n-1}^{(2)} = 0,$$

and that $h_{i \rightarrow j}^{(k)} \neq 0$ as far as (i, j, k) satisfies the condition (iii) in Proposition 5.14. Hence we have shown Proposition 5.14. \square

We shall use the basis $h_{i \rightarrow i-1}^{(k)}$ in Chapter 6, and $h_{i \rightarrow i+1}^{(1)}$ in Chapter 7, respectively. For later purpose, we give explicit formulæ of $h_{i \rightarrow j}^{(k)}(e_I)$ for $I \in \mathcal{I}_{n,i}$ in Table 5.1. The proof is immediate from (5.7)–(5.11) and the definitions (5.22)–(5.27). Here we recall from (5.12) that $\tilde{Q}_J(\zeta') = \sum_{\ell \in J} \zeta_\ell^2 - \frac{i}{n-1} Q_{n-1}(\zeta')$ for $\zeta' = (\zeta_1, \dots, \zeta_{n-1})$ and $J \in \mathcal{I}_{n-1,i}$.

TABLE 5.1. Formulæ of $h_{i \rightarrow j}^{(k)}(e_I)$ for $I \in \mathcal{I}_{n,i}$

	$n \notin I$	$n \in I$
$h_{i \rightarrow i-1}^{(0)}(e_I)$	0	$(-1)^{i-1} e_{I \setminus \{n\}}$
$h_{i \rightarrow i}^{(0)}(e_I)$	e_I	0
$h_{i \rightarrow i-2}^{(1)}(e_I)$	0	$-\sum_{\ell \in I \setminus \{n\}} \text{sgn}(I; \ell, n) e_{I \setminus \{\ell, n\}} \zeta_\ell$
$h_{i \rightarrow i-1}^{(1)}(e_I)$	$\sum_{\ell \in I} \text{sgn}(I; \ell) e_{I \setminus \{\ell\}} \zeta_\ell$	0
$h_{i \rightarrow i}^{(1)}(e_I)$	0	$\sum_{\ell \notin I} \text{sgn}(I; \ell, n) e_{I \setminus \{n\} \cup \{\ell\}} \zeta_\ell$
$h_{i \rightarrow i+1}^{(1)}(e_I)$	$\sum_{\substack{\ell \notin I \\ \ell \neq n}} \text{sgn}(I; \ell) e_{I \cup \{\ell\}} \zeta_\ell$	0

$$h_{i \rightarrow i-1}^{(2)}(e_I) = \begin{cases} 0 & \text{if } n \notin I, \\ (-1)^{i-1} \left(\tilde{Q}_{I \setminus \{n\}}(\zeta') e_{I \setminus \{n\}} + \sum_{p \in I \setminus \{n\}} \sum_{q \notin I} \text{sgn}(I; p, q) \zeta_p \zeta_q e_{I \setminus \{p, n\} \cup \{q\}} \right) & \text{if } n \in I. \end{cases}$$

$$h_{i \rightarrow i}^{(2)}(e_I) = \begin{cases} \tilde{Q}_I(\zeta') e_I + \sum_{p \in I} \sum_{\substack{q \notin I \\ q \neq n}} \text{sgn}(I; p, q) \zeta_p \zeta_q e_{I \setminus \{p\} \cup \{q\}} & \text{if } n \notin I, \\ 0 & \text{if } n \in I. \end{cases}$$

5.6. Descriptions of $\text{Hom}_{O(n-1)}(\bigwedge^i(\mathbb{C}^n), \bigwedge^j(\mathbb{C}^{n-1}) \otimes \text{Pol}[\zeta_1, \dots, \zeta_n])$.

It follows from Lemma 4.2 and Proposition 5.14 that the spaces $\text{Hom}_{O(n-1)}(V, W \otimes \text{Pol}(\mathfrak{n}_+))$ are determined explicitly for $(V, W) = (\bigwedge^i(\mathbb{C}^n), \bigwedge^j(\mathbb{C}^{n-1}))$ as follows:

Proposition 5.16. *Let $n \geq 2$, $0 \leq i \leq n$, and $0 \leq j \leq n-1$. Then the following two conditions on (i, j) are equivalent:*

- (i) $\text{Hom}_{O(n-1)}(\bigwedge^i(\mathbb{C}^n), \bigwedge^j(\mathbb{C}^{n-1}) \otimes \text{Pol}[\zeta_1, \dots, \zeta_n]) \neq \{0\}$.
- (ii) $j \in \{i-2, i-1, i, i+1\}$.

Proposition 5.17. *Let $n \geq 2$. Suppose that $0 \leq i \leq n$, $0 \leq j \leq n-1$, and $a \in \mathbb{N}$. Then $\text{Hom}_{O(n-1)}(\bigwedge^i(\mathbb{C}^n), \bigwedge^j(\mathbb{C}^{n-1}) \otimes \text{Pol}^a[\zeta_1, \dots, \zeta_n])$ is equal to*

$$\begin{aligned} & \{(T_{a-1}g)h_{i \rightarrow i-2}^{(1)} : g \in \text{Pol}_{a-1}[t]_{\text{even}}\} && \text{if } j = i-2, \\ & \mathbb{C}\text{-span} \left\{ (T_{a-k}g_k)h_{i \rightarrow i-1}^{(k)} : g_k \in \text{Pol}_{a-k}[t]_{\text{even}} \right\} && \text{if } j = i-1, \\ & \mathbb{C}\text{-span} \left\{ (T_{a-k}g_k)h_{i \rightarrow i}^{(k)} : g_k \in \text{Pol}_{a-k}[t]_{\text{even}} \right\} && \text{if } j = i, \\ & \left\{ (T_{a-1}g)h_{i \rightarrow i+1}^{(1)} : g \in \text{Pol}_{a-1}[t]_{\text{even}} \right\} && \text{if } j = i+1, \\ & \{0\} && \text{otherwise.} \end{aligned}$$

Here we regard $\text{Pol}_{-1}[t]_{\text{even}} = \{0\}$. We also regard

$$(5.29) \quad h_{i \rightarrow i-1}^{(k)} = 0 \quad \text{when } (i, k) = (1, 2), (n, 1), \text{ or } (n, 2),$$

$$(5.30) \quad h_{i \rightarrow i}^{(k)} = 0 \quad \text{when } (i, k) = (0, 1), (0, 2), \text{ or } (n-1, 2).$$

We note that when $j = i, i-1$, we have $k \leq \min\{2, a\}$.

5.7. Proof of the implication (i) \Rightarrow (iii) in Theorem 2.8. In this section, we give a proof of the implication (i) \Rightarrow (iii) in Theorem 2.8.

We recall that characters of A are parametrized by \mathbb{C} via the normalization (2.6). For $0 \leq i \leq n$, $\alpha \in \mathbb{Z}/2\mathbb{Z}$, and $\lambda \in \mathbb{C}$, we denote by $\sigma_{\lambda, \alpha}^{(i)}$ the outer tensor product representation $\bigwedge^i(\mathbb{C}^n) \boxtimes (-1)^\alpha \boxtimes \mathbb{C}_\lambda$ of the Levi subgroup $L = MA \simeq O(n) \times O(1) \times \mathbb{R}$ on the i -th exterior tensor space $\bigwedge^i(\mathbb{C}^n)$. Similarly, $\tau_{\nu, \beta}^{(j)}$ ($0 \leq j \leq n-1$, $\nu \in \mathbb{C}$, $\beta \in \mathbb{Z}/2\mathbb{Z}$) stands for the outer tensor product representation $\bigwedge^j(\mathbb{C}^{n-1}) \boxtimes (-1)^\beta \boxtimes \mathbb{C}_\nu$ of the Levi subgroup $L' = M'A \simeq O(n-1) \times O(1) \times \mathbb{R}$.

Lemma 5.18. *Suppose that $n \geq 2$. Let $0 \leq i \leq n$, $0 \leq j \leq n-1$, $\lambda, \nu \in \mathbb{C}$, $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$ and $a \in \mathbb{N}$. Then the following two conditions on $(i, j, \lambda, \nu, \alpha, \beta, a)$ are equivalent:*

- (i) $\text{Hom}_{L'}(\sigma_{\lambda, \alpha}^{(i)}|_{L'}, \tau_{\nu, \beta}^{(j)} \otimes \text{Pol}^a(\mathfrak{n}_+)) \neq \{0\}$.
- (ii) $j \in \{i-2, i-1, i, i+1\}$, $\nu - \lambda = a$, and $\beta - \alpha \equiv a \pmod{2}$.
Moreover, $a \geq 1$ when $j = i-2$ or $i+1$.

Proof. First of all, we consider the actions of the second and third factors of $L' \simeq O(n-1) \times O(1) \times \mathbb{R}$. Since $e^{tH_0} \in A$ and $-1 \in O(1)$ act on $\mathfrak{n}_+ \simeq \mathbb{C}^n$ as the scalars e^t and -1 , respectively,

$$\text{Hom}_{O(1) \times A}(\sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(j)} \otimes \text{Pol}^a(\mathfrak{n}_+)) \neq \{0\}$$

if and only if

$$\nu = \lambda + a \quad \text{and} \quad \beta \equiv \alpha + a \pmod{2}.$$

Then the proof of the lemma reduces to Proposition 5.17 for the action of the first factor $O(n-1)$ of L' . \square

We recall from Section 2.1 that $I(i, \lambda)_\alpha$ and $J(j, \nu)_\beta$ are (unnormalized) principal series representations of G and G' , respectively. By the F-method summarized in Fact 3.3, we have a natural bijection

$$(5.31) \quad \text{Diff}_{G'}(I(i, \lambda)_\alpha, J(j, \nu)_\beta) \simeq \text{Sol}(\mathfrak{n}_+; \sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(j)}),$$

where we recall from (3.5)

$$\text{Sol}(\mathfrak{n}_+; \sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(j)}) = \{\psi \in \text{Hom}_{L'}(\sigma_{\lambda, \alpha}^{(i)}|_{L'}, \tau_{\nu, \beta}^{(j)} \otimes \text{Pol}(\mathfrak{n}_+)) : \widehat{d\pi_{(i, \lambda)}^*}(C)\psi = 0 \text{ for all } C \in \mathfrak{n}'_+\}.$$

Proposition 5.19. *Suppose $0 \leq i \leq n$, $0 \leq j \leq n-1$, $\lambda, \nu \in \mathbb{C}$, and $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$. Then,*

(1) $\text{Diff}_{G'}(I(i, \lambda)_\alpha, J(j, \nu)_\beta) \neq \{0\}$ *only if*

$$(5.32) \quad j \in \{i-2, i-1, i, i+1\}, \quad \nu - \lambda \in \mathbb{N}, \quad \text{and} \quad \beta - \alpha \equiv \nu - \lambda \pmod{2}.$$

(2) *Suppose (5.32) is satisfied. Then,*

$$\text{Sol}(\mathfrak{n}_+; \sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(j)}) = \left\{ \psi \in \text{Hom}_{O(n-1)}(\wedge^i(\mathbb{C}^n), \wedge^j(\mathbb{C}^{n-1}) \otimes \text{Pol}^{\nu-\lambda}(\mathfrak{n}_+)) : \widehat{d\pi_{(i, \lambda)}^*}(N_1^+)\psi = 0 \right\}.$$

Proof. (1) The first assertion is a direct consequence of (5.31) and Lemma 5.18.

(2) Suppose (5.32) is fulfilled. Then, it follows from the proof of Lemma 5.18 that

$$\text{Hom}_{L'}(\sigma_{\lambda, \alpha}^{(i)}|_{L'}, \tau_{\nu, \beta}^{(j)} \otimes \text{Pol}(\mathfrak{n}_+)) \simeq \text{Hom}_{O(n-1)}(\wedge^i(\mathbb{C}^n), \wedge^j(\mathbb{C}^{n-1}) \otimes \text{Pol}^{\nu-\lambda}(\mathfrak{n}_+)).$$

Hence the second statement follows. \square

Owing to Proposition 5.17 and Proposition 5.19 (2), the F-system for $\text{Sol}(\mathfrak{n}_+; \sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(j)})$ boils down to a system of ordinary differential equations of $g_j(t)$ ($j = 0, 1, 2$). We shall find explicitly the polynomials $g_j(t)$, and determine $\text{Sol}(\mathfrak{n}_+; \sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(j)})$ for $j = i-1$ in Chapter 6, and for $j = i+1$ in Chapter 7.

6. F-SYSTEM FOR SYMMETRY BREAKING OPERATORS ($j = i - 1, i$ CASE)

As we discussed in the previous chapter, the F-method (see Fact 3.3) establishes a natural bijection (5.31) between the space $\text{Diff}_{G'}(I(i, \lambda)_\alpha, J(j, \nu)_\beta)$ of differential symmetry breaking operators and the space $\text{Sol}(\mathfrak{n}_+; \sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(j)})$ of $\text{Hom}_{\mathbb{C}}(\bigwedge^i(\mathbb{C}^n), \bigwedge^j(\mathbb{C}^{n-1}))$ -valued polynomial solutions to the F-system.

In this chapter, we study the F-system in detail for $j = i - 1$. The case $j = i + 1$ will be investigated in the next chapter. Via the duality theorem (see Theorem 2.7), the cases $j = i, i - 2$ are understood as the dual to the cases $j = i - 1, i + 1$, respectively. The results of this chapter ($j = i - 1$ case) are summarized as follows. We recall from (5.25) that $h_{i \rightarrow i-1}^{(k)}: \bigwedge^i(\mathbb{C}^n) \rightarrow \bigwedge^{i-1}(\mathbb{C}^{n-1}) \otimes \mathcal{H}^k(\mathbb{C}^{n-1})$ are $O(n-1)$ -homomorphisms for $k = 0, 1$, and 2 . Let $\tilde{C}_\ell^\mu(t)$ be the renormalized Gegenbauer polynomial (see (14.3)).

Theorem 6.1. *Let $n \geq 3$. Suppose $1 \leq i \leq n$, $\lambda, \nu \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$. Let $\sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(i-1)}$ be the outer tensor product representations of $L = MA \simeq O(n) \times O(1) \times \mathbb{R}$, $L' = M'A \simeq O(n-1) \times O(1) \times \mathbb{R}$ on $\bigwedge^i(\mathbb{C}^n) \boxtimes (-1)^\alpha \boxtimes \mathbb{C}_\lambda$, $\bigwedge^{i-1}(\mathbb{C}^{n-1}) \boxtimes (-1)^\beta \boxtimes \mathbb{C}_\nu$, respectively. Then*

(6.1)

$$\text{Sol}\left(\mathfrak{n}_+; \sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(i-1)}\right) = \begin{cases} \mathbb{C} \sum_{k=0}^2 (T_{\nu-\lambda-k} g_k) h_{i \rightarrow i-1}^{(k)} & \text{if } \nu - \lambda \in \mathbb{N} \text{ and } \beta - \alpha \equiv \nu - \lambda \pmod{2}, \\ \{0\} & \text{otherwise.} \end{cases}$$

From now, we assume $a := \nu - \lambda \in \mathbb{N}$ and $\beta - \alpha \equiv a \pmod{2}$. We consider the following polynomials:

$$(6.2) \quad e^{-\frac{\pi\sqrt{-1}}{2}} B t \tilde{C}_{a-1}^{\lambda-\frac{n-3}{2}}\left(e^{\frac{\pi\sqrt{-1}}{2}} t\right) + C \tilde{C}_{a-2}^{\lambda-\frac{n-3}{2}}\left(e^{\frac{\pi\sqrt{-1}}{2}} t\right),$$

$$(6.3) \quad e^{-\frac{\pi\sqrt{-1}}{2}} A \tilde{C}_{a-1}^{\lambda-\frac{n-3}{2}}\left(e^{\frac{\pi\sqrt{-1}}{2}} t\right),$$

$$(6.4) \quad \tilde{C}_{a-2}^{\lambda-\frac{n-3}{2}}\left(e^{\frac{\pi\sqrt{-1}}{2}} t\right),$$

where

$$A = \gamma\left(\lambda - \frac{n-1}{2}, a\right), \quad B = A \left(1 + \frac{\lambda - n + i}{a}\right), \quad C = \frac{\lambda - n + i}{a} + \frac{i-1}{n-1}$$

with $\gamma(\mu, a) = 1$ (a : odd); $= \mu + \frac{a}{2}$ (a : even) (see (1.3)). Then the polynomials $g_k(t)$ ($k = 0, 1, 2$) are given as follows.

- (1) $i = 1, a \geq 1 :$ $(g_0(t), g_1(t), g_2(t)) = ((6.2), (6.3), 0);$
- (2) $2 \leq i \leq n-1, a \geq 1 :$ $(g_0(t), g_1(t), g_2(t)) = ((6.2), (6.3), (6.4));$
- (3) $i = n, a \geq 1 :$ $(g_0(t), g_1(t), g_2(t)) = \left(\tilde{C}_a^{\lambda - \frac{n-3}{2}} \left(e^{\frac{\pi\sqrt{-1}}{2}t} \right), 0, 0 \right);$
- (4) $1 \leq i \leq n, a = 0 :$ $(g_0(t), g_1(t), g_2(t)) = (1, 0, 0).$

Remark 6.2. The exceptional cases (3) and (4) in Theorem 6.1 are closely related to the vanishing conditions of the family of the symmetry breaking operators $\mathcal{D}_{u,a}^{i \rightarrow i-1}$ given in Proposition 1.4. As we introduced the renormalized operator $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i-1}$ in (1.9), we separated (3) and (4) from the others. The relationship will be clarified in Section 10.1 where we explain how the triple (g_0, g_1, g_2) of polynomials gives rise to the differential symmetry breaking operator $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i-1} (= \tilde{\mathcal{C}}_{\lambda,\nu}^{i,i-1})$ from $\mathcal{E}^i(\mathbb{R}^n)$ to $\mathcal{E}^{i-1}(\mathbb{R}^{n-1})$ as stated in Theorem 2.9.

The proof of Theorem 6.1 is divided into two parts:

- to reduce a system of partial differential equations (F-system) to a system of ordinary differential equations on $g_k(t)$ ($k = 0, 1, 2$) (see Theorem 6.5).
- to find explicit polynomial solutions $\{g_k(t)\}$ to the latter system (see Theorem 6.7).

In the next section we first complete the proof of Theorem 2.8 for $j = i-1, i$ by admitting Theorem 6.1.

6.1. Proof of Theorem 2.8 for $j = i-1, i$. In this section, we prove that Theorem 6.1 determines the dimension of the space of differential symmetry breaking operators from principal series representations $I(i, \lambda)_\alpha$ of $G = O(n+1, 1)$ to $J(j, \nu)_\beta$ of $G' = O(n, 1)$ when $j = i-1, i$. The following two theorems correspond to Theorem 2.8 in the cases $j = i-1$ and i , respectively.

Theorem 6.3 ($j = i-1$ case). *Let $n \geq 3$. Suppose $1 \leq i \leq n$, $\lambda, \nu \in \mathbb{C}$, and $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$. Then the following three conditions on $(i, \lambda, \nu, \alpha, \beta)$ are equivalent:*

- (i) $\text{Diff}_{G'}(I(i, \lambda)_\alpha, J(i-1, \nu)_\beta) \neq \{0\}.$
- (ii) $\dim \text{Diff}_{G'}(I(i, \lambda)_\alpha, J(i-1, \nu)_\beta) = 1.$
- (iii) $\nu - \lambda \in \mathbb{N}, \alpha - \beta \equiv \nu - \lambda \pmod{2}.$

Theorem 6.4 ($j = i$ case). *Let $n \geq 3$. Suppose $0 \leq i \leq n-1$, $\lambda, \nu \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$. Then the following three conditions on $(i, \lambda, \nu, \alpha, \beta)$ are equivalent:*

- (i) $\text{Diff}_{G'}(I(i, \lambda)_\alpha, J(i, \nu)_\beta) \neq \{0\}.$
- (ii) $\dim \text{Diff}_{G'}(I(i, \lambda)_\alpha, J(i, \nu)_\beta) = 1.$
- (iii) $\nu - \lambda \in \mathbb{N}, \alpha - \beta \equiv \nu - \lambda \pmod{2}.$

Proof of Theorem 6.3. By the general theory of the F-method (see (5.31)), we have the vector space isomorphism (5.31). Thus the equivalence will follow if we show that the solutions in Theorem 6.1 are nonzero when the condition (iii) is satisfied.

For this, we observe that we renormalized the Gegenbauer polynomial in a way that $\tilde{C}_\ell^\alpha(t)$ is nonzero for all $\alpha \in \mathbb{C}$ and $\ell \in \mathbb{N}$ (see Section 14.1). We also know that $h_{i \rightarrow i-1}^{(k)} \neq 0$ except for the cases $(i, k) = (1, 2), (n, 1)$, or $(n, 2)$ (see (5.29)). Moreover, for each i ($1 \leq i \leq n$), these nonzero maps $h_{i \rightarrow i-1}^{(k)}$ ($k = 0, 1, 2$) are linearly independent by Proposition 5.14. Hence the solutions constructed in Theorem 6.1 are nonzero by the decomposition (4.3). Now the desired statement is proved. \square

For the proof of Theorem 6.4, we use the duality theorem for symmetry breaking operators (Theorem 2.7) instead of solving the F-system.

Proof of Theorem 6.4. It follows from Theorem 2.7 that we have a natural bijection

$$\text{Diff}_{G'}(I(i, \lambda)_\alpha, J(i, \nu)_\beta) \simeq \text{Diff}_{G'}(I(\tilde{i}, \lambda)_\alpha, J(\tilde{i} - 1, \nu)_\beta),$$

where $\tilde{i} := n - i$. Then it is easy to see that $(\tilde{i}, \lambda, \nu, \alpha, \beta)$ satisfies the condition (iii) in Theorem 6.3 if and only if $(i, \lambda, \nu, \alpha, \beta)$ satisfies (iii) in Theorem 6.4. Hence Theorem 6.4 is deduced from Theorem 6.3. \square

6.2. Reduction theorem. We begin by stating the main theorem of the rest of this chapter. Recall from Section 4.4 that, for $\mu \in \mathbb{C}$ and $\ell \in \mathbb{N}$, R_ℓ^μ denotes the following differential operator

$$R_\ell^\mu = -\frac{1}{2} \left((1 + t^2) \frac{d^2}{dt^2} + (1 + 2\mu)t \frac{d}{dt} - \ell(\ell + 2\mu) \right),$$

equivalently,

$$(6.5) \quad R_\ell^\mu = -\frac{1}{2t^2} \left((1 + t^2)\vartheta_t^2 - (1 - 2\mu t^2)\vartheta_t - \ell(\ell + 2\mu)t^2 \right)$$

with $\vartheta_t := t \frac{d}{dt}$. For polynomials $g_j(t)$ ($j = 0, 1, 2$) of one variable t , we then define other polynomials $L_r(g_0, g_1, g_2)(t)$ of the same variable t ($r = 1, 2, \dots, 7$) as follows:

$$(6.6) \quad L_1(g_0, g_1, g_2) := R_{a-2}^{\lambda - \frac{n-3}{2}} g_2,$$

$$(6.7) \quad L_2(g_0, g_1, g_2) := R_{a-1}^{\lambda - \frac{n-3}{2}} g_1,$$

$$(6.8) \quad L_3(g_0, g_1, g_2) := (a - 1 - \vartheta_t)g_1 - \frac{dg_2}{dt},$$

$$(6.9) \quad L_4(g_0, g_1, g_2) := (\vartheta_t + 2\lambda + a - n + 1)g_2 - \frac{dg_1}{dt},$$

$$(6.10) \quad L_5(g_0, g_1, g_2) := \frac{dg_0}{dt} + \frac{n-i}{n-1} \frac{dg_2}{dt} - (a + \lambda - n + i)g_1,$$

$$(6.11) \quad L_6(g_0, g_1, g_2) := (\lambda - n + i + a - \frac{n-i}{n-1}(a - \vartheta_t))g_2 - (a - \vartheta_t)g_0,$$

$$(6.12) \quad L_7(g_0, g_1, g_2) := R_a^{\lambda - \frac{n-1}{2}} g_0 + \frac{n-i}{n-1} \frac{dg_1}{dt}.$$

For later convenience we also set $L_8(g_0, g_1, g_2) := L_6(g_0, g_1, g_2) - tL_5(g_0, g_1, g_2)$, $L_9(g_0, g_1, g_2) := \frac{n-i}{n-1}L_3(g_0, g_1, g_2) + L_5(g_0, g_1, g_2)$, namely,

$$(6.13) \quad \begin{aligned} L_8(g_0, g_1, g_2) &= \left(\lambda - n + i + \frac{a(i-1)}{n-1} \right) g_2(t) + (a + \lambda - n + i)t g_1(t) - a g_0(t), \\ L_9(g_0, g_1, g_2) &= \frac{dg_0}{dt} + \left(\frac{i-1}{n-1}(\vartheta_t - n - a + 2) - (\vartheta_t + \lambda - n + 2) \right) g_1. \end{aligned}$$

Note that $L_1(g_0, g_1, g_2), \dots, L_4(g_0, g_1, g_2)$ are independent of g_0 . Likewise, $L_2(g_0, g_1, g_2), L_7(g_0, g_1, g_2)$, and $L_9(g_0, g_1, g_2)$ are independent of g_2 .

By Proposition 5.17, any element $\psi \in \text{Hom}_{O(n-1)}(\bigwedge^i(\mathbb{C}^n), \bigwedge^{i-1}(\mathbb{C}^{n-1}) \otimes \text{Pol}^a[\zeta_1, \dots, \zeta_n])$ is of the form

$$\psi = \begin{cases} \sum_{k=0}^1 (T_{a-k}g_k)h_{1 \rightarrow 0}^{(k)} & (i = 1), \\ \sum_{k=0}^2 (T_{a-k}g_k)h_{i \rightarrow i-1}^{(k)} & (2 \leq i \leq n-1), \\ (T_a g_0)h_{n \rightarrow n-1}^{(0)} & (i = n), \end{cases}$$

for some polynomials $g_k(t) \in \text{Pol}_{a-k}[t]_{\text{even}}$ ($k = 0, 1, 2$), where $T_{a-k}g_k \in \text{Pol}^{a-k}[\zeta_1, \dots, \zeta_n]$ are given as in (4.4). For $i = 1$ or n , we may also write as $\psi = \sum_{k=0}^2 (T_{a-k}g_k)h_{i \rightarrow i-1}^{(k)}$ with $g_2 = 0$ for $i = 1$ and $g_1 = g_2 = 0$ for $i = n$. In what follows, we understand

$$(6.14) \quad g_1 = g_2 = 0 \quad \text{for } a = 0; \quad g_2 = 0 \quad \text{for } a = 1; \quad g_2 = 0 \quad \text{for } i = 1; \quad g_1 = g_2 = 0 \quad \text{for } i = n.$$

Theorem 6.1 can be separated into Theorem 6.5 (finding equations) and Theorem 6.7 (finding solutions) below.

Theorem 6.5. *Let $n \geq 3$ and $1 \leq i \leq n$. Then, for $\psi = \sum_{k=0}^2 (T_{a-k} g_k) h_{i \rightarrow i-1}^{(k)}$, the following hold.*

- (1) *Suppose $i = 1$. The following two conditions on g_0, g_1 are equivalent:*
 - (i) *ψ satisfies $\widehat{d\pi_{(1,\lambda)}^*}(C)\psi = 0$ for all $C \in \mathfrak{n}'_+$.*
 - (ii) *$L_r(g_0, g_1, 0) = 0$ for $r = 2, 7, 9$.*
- (2) *Suppose $2 \leq i \leq n-1$. The following two conditions on g_0, g_1, g_2 are equivalent:*
 - (i) *ψ satisfies $\widehat{d\pi_{(i,\lambda)}^*}(C)\psi = 0$ for all $C \in \mathfrak{n}'_+$.*
 - (ii) *$L_r(g_0, g_1, g_2) = 0$ for all $r = 1, \dots, 7$.*
- (3) *Suppose $i = n$. The following two conditions on g_0 are equivalent:*
 - (i) *ψ satisfies $\widehat{d\pi_{(n,\lambda)}^*}(C)\psi = 0$ for all $C \in \mathfrak{n}'_+$.*
 - (ii) *$L_7(g_0, 0, 0) = 0$.*

Remark 6.6. For $i = n$, the equation $L_7(g_0, 0, 0) = 0$ amounts to the imaginary Gegenbauer differential equation $R_a^{\lambda - \frac{n-1}{2}} g_0 = 0$.

Theorem 6.7. *Let $n \geq 3$ and $1 \leq i \leq n$. Suppose $g_k(t) \in \text{Pol}_{a-k}[t]_{\text{even}}$ ($k = 0, 1, 2$) with the convention (6.14). Then, up to scalar multiple, the solution (g_0, g_1, g_2) of the F-system $L_r(g_0, g_1, 0) = 0$ for $r = 2, 7, 9$ when $i = 1$; $L_r(g_0, g_1, g_2) = 0$ for $r = 1, \dots, 7$ when $2 \leq i \leq n-1$; $L_r(g_0, 0, 0) = 0$ for $r = 7$ when $i = n$, is given as follows:*

- (1) $i = 1, a \geq 1 :$ $(g_0, g_1, g_2) = ((6.2), (6.3), 0);$
- (2) $2 \leq i \leq n-1, a \geq 1 :$ $(g_0, g_1, g_2) = ((6.2), (6.3), (6.4));$
- (3) $i = n, a \geq 1 :$ $(g_0, g_1, g_2) = \left(\tilde{C}_a^{\lambda - \frac{n-1}{2}} \left(e^{\frac{\pi\sqrt{-1}}{2}t} \right), 0, 0 \right);$
- (4) $1 \leq i \leq n, a = 0 :$ $(g_0, g_1, g_2) = (1, 0, 0).$

Remark 6.8. The formula (3) for $a = 0$ coincides with the formula (4) for $i = n$ because $\tilde{C}_0^\mu(t) = 1$.

The proof of Theorem 6.7 will be given in Section 14.5 by using some basic properties of the Gegenbauer polynomials that are summarized in Appendix. Alternatively, the theorem could also be shown by solving directly the F-system $L_r(g_0, g_1, g_2) = 0$ ($r = 1, \dots, 7$) with the following remark.

Remark 6.9. Let $a \geq 3$ and assume that (g_0, g_1, g_2) satisfies $L_r(g_0, g_1, g_2) = 0$ for $r = 1, 2, 3$. Then the following two conditions on the triple (g_0, g_1, g_2) are equivalent:

- (i) $L_r(g_0, g_1, g_2) = 0$ for $r = 4, 5, 6, 7$.
- (ii) $L_8(g_0, g_1, g_2) = 0$.

The rest of this chapter is devoted to proving Theorem 6.5. Since the argument requires a number of lemmas and propositions, we separate it into a few steps as follows. Let N_1^+ be the element of the nilpotent Lie algebra \mathfrak{n}'_+ defined in (2.2).

- Step 1. Reduce the condition $\widehat{d\pi_{(i,\lambda)^*}}(C)\psi = 0$ for all $C \in \mathfrak{n}'_+$ to $\widehat{d\pi_{(i,\lambda)^*}}(N_1^+)\psi = 0$.
- Step 2. Consider the equation $\widehat{d\pi_{(i,\lambda)^*}}(N_1^+)\psi = 0$ in terms of matrix coefficients M_{IJ} .
- Step 3. Reduce the number of cases for the matrix coefficients M_{IJ} to consider.
- Step 4. Express the matrix coefficients M_{IJ} in terms of $L_r(g_0, g_1, g_2)$ for $r = 1, \dots, 7$.
- Step 5. Deduce $L_r(g_0, g_1, g_2) = 0$ for $r = 1, \dots, 7$ (resp. for $r = 7$) from $M_{IJ} = 0$ for $2 \leq i \leq n-1$ (resp. for $i = n$)

Observe that Step 1 was done in Lemma 3.4 in a more general setting (see also Proposition 5.19 (2)). In the next sections we shall discuss Steps 2–5.

6.3. Step 2: Matrix coefficients M_{IJ} for $\widehat{d\pi_{(i,\lambda)^*}}(N_1^+)\psi$. In this section, along the strategy discussed in Section 4.5, we consider the differential equation (F-system) $\widehat{d\pi_{(i,\lambda)^*}}(N_1^+)\psi = 0$ in terms of matrix coefficients $M_{IJ} = M_{IJ}^{\text{scalar}} + M_{IJ}^{\text{vect}}$. The scalar part M_{IJ}^{scalar} of M_{IJ} is also computed. Since the arguments work for any n and i , we assume that $n \geq 1$ and $1 \leq i \leq n$ in this section.

We begin with a quick review of Section 4.5. First, for $\ell \in \{1, \dots, n\}$ and $m \in \{0, 1, \dots, n\}$, we write

$$\mathcal{I}_{\ell, m} = \{R \subset \{1, \dots, \ell\} : |R| = m\}$$

as in (5.1). Here $\mathcal{I}_{\ell, 0}$ is understood as $\mathcal{I}_{\ell, 0} = \{\emptyset\}$. Note that $\{e_I : I \in \mathcal{I}_{n, i}\}$ and $\{w_J : J \in \mathcal{I}_{n-1, i-1}\}$ are the standard bases of $\bigwedge^i(\mathbb{C}^n)$ and $\bigwedge^{i-1}(\mathbb{C}^{n-1})$, respectively. For $\{e_I : I \in \mathcal{I}_{n, i}\}$ and $\{w_J : J \in \mathcal{I}_{n-1, i-1}\}$, we then set

$$M_{IJ} \equiv M_{IJ}(g_0, g_1, g_2) := \left\langle \widehat{d\pi_{(i,\lambda)^*}}(N_1^+)\psi(\zeta) e_I, w_J^\vee \right\rangle.$$

Lemma 6.10. *The following two conditions on (g_0, g_1, g_2) are equivalent:*

- (i) $\widehat{d\pi_{(i,\lambda)^*}}(N_1^+)\psi = 0$.
- (ii) $M_{IJ} = 0$ for all $I \in \mathcal{I}_{n, i}$ and $J \in \mathcal{I}_{n-1, i-1}$.

Proof. Clear. □

According to the decomposition (3.8) into the “scalar part” and “vector part”

$$\widehat{d\pi_{(i,\lambda)^*}}(N_1^+) = \widehat{d\pi_{\lambda^*}}(N_1^+) \otimes \text{id}_{V^\vee} + A_\sigma(N_1^+)$$

with $V = \bigwedge^i(\mathbb{C}^n)$, we decompose M_{IJ} as

$$M_{IJ} = M_{IJ}^{\text{scalar}} + M_{IJ}^{\text{vect}}$$

(see Proposition 4.9).

For $I \in \mathcal{I}_{n,i}$ and $J \in \mathcal{I}_{n-1,i-1}$, we write $h_{IJ}^{(k)}$ for the matrix coefficient $\left(h_{i \rightarrow i-1}^{(k)}\right)_{IJ} = \langle h_{i \rightarrow i-1}^{(k)}(e_I), e_J^\vee \rangle$ of $h_{i \rightarrow i-1}^{(k)}$. It follows from Table 5.1 that we have

$$(6.15) \quad (-1)^{i-1} h_{IJ}^{(0)} = \begin{cases} 1 & \text{if } J \subset I \ni n, \\ 0 & \text{otherwise.} \end{cases}$$

$$(6.16) \quad h_{IJ}^{(1)} = \begin{cases} \text{sgn}(I; \ell) \zeta_\ell & \text{if } J \subset I \not\ni n, \\ 0 & \text{otherwise.} \end{cases}$$

$$(6.17) \quad (-1)^{i-1} h_{IJ}^{(2)} = \begin{cases} \sum_{\ell \in I \setminus \{n\}} \zeta_\ell^2 - \frac{i-1}{n-1} Q_{n-1}(\zeta') & \text{if } J \subset I \ni n, \\ \text{sgn}(I; p, q) \zeta_p \zeta_q & \text{if } |J \setminus I| = 1 \text{ and } I \ni n, \\ 0 & \text{otherwise.} \end{cases}$$

Here $Q_{n-1}(\zeta') = \sum_{m=1}^{n-1} \zeta_m^2$, and we write $I = J \cup \{\ell\}$ ($1 \leq \ell \leq n-1$) if $J \subset I \not\ni n$, and $I = K \cup \{p, n\}$, $J = K \cup \{q\}$ if $|J \setminus I| = 1$ and $I \ni n$. By (6.15)-(6.17), we observe:

$$(6.18) \quad h_{IJ}^{(k)} = 0 \quad \text{for } n \notin I \text{ (i.e. } I \in \mathcal{I}_{n-1,i} \text{) for } k = 0, 2,$$

$$(6.19) \quad h_{IJ}^{(1)} = 0 \quad \text{for } n \in I \text{ (i.e. } I \in \mathcal{I}_{n,i} \setminus \mathcal{I}_{n-1,i} \text{)}.$$

By using $h_{IJ}^{(k)}$, we then have the following.

Lemma 6.11. *With $G_k := T_{a-k} \left(R_{a-k}^{\lambda - \frac{n-1}{2}} g_k \right)$ for $k = 0, 1, 2$, the scalar part M_{IJ}^{scalar} is given as follows.*

$$M_{IJ}^{\text{scalar}} = \begin{cases} \frac{\zeta_1}{Q_{n-1}(\zeta')} G_0 h_{IJ}^{(0)} + \frac{\zeta_1}{Q_{n-1}(\zeta')} G_2 h_{IJ}^{(2)} + (\lambda + a - 1) T_{a-2} g_2 \frac{\partial h_{IJ}^{(2)}}{\partial \zeta_1} & (n \in I), \\ \frac{\zeta_1}{Q_{n-1}(\zeta')} G_1 h_{IJ}^{(1)} + (\lambda + a - 1) T_{a-1} g_1 \frac{\partial h_{IJ}^{(1)}}{\partial \zeta_1} & (n \notin I). \end{cases}$$

Proof. As $\psi = \sum_{k=0}^2 (T_{a-k} g_k) h_{i \rightarrow i-1}^{(k)}$, it follows from Proposition 4.4 (1) that

$$M_{IJ}^{\text{scalar}} = \sum_{k=0}^2 \left(\frac{\zeta_1}{Q_{n-1}(\zeta')} G_k h_{IJ}^{(k)} + (\lambda + a - 1) (T_{a-k} g_k) \frac{\partial h_{IJ}^{(k)}}{\partial \zeta_1} \right).$$

Since $\frac{\partial h_{IJ}^{(0)}}{\partial \zeta_1} = 0$ by (6.15), the proposed identity holds from (6.18) and (6.19). \square

In Section 6.6, by using (6.15)-(6.17) and Lemma 6.11, we shall give explicit formulæ for $M_{IJ} = M_{IJ}^{\text{scalar}} + M_{IJ}^{\text{vect}}$.

We conclude this section by showing the following lemma.

Lemma 6.12. *The following hold.*

- (1) *If $|J \setminus I| \geq 1$ and $n \notin I$, then $M_{IJ}^{\text{scalar}} = 0$.*
- (2) *If $|J \setminus I| \geq 2$ and $n \in I$, then $M_{IJ}^{\text{scalar}} = 0$.*

Consequently, if $|J \setminus I| \geq 2$, then $M_{IJ}^{\text{scalar}} = 0$.

Proof. To show the first claim, as $n \notin I$, it suffices to show that $h_{IJ}^{(1)} = 0$. Since $J \not\subset I$, it follows that $h_{IJ}^{(1)} = 0$. The second claim may be shown similarly. Indeed, if $|J \setminus I| \geq 2$, then $h_{IJ}^{(k)} = 0$ for $k = 0, 1, 2$. Therefore the second claim also holds. \square

The vector part M_{IJ}^{scalar} will be treated in the next section.

6.4. Step 3: Case-reduction for M_{IJ}^{vect} . In view of Lemma 6.10, we wish to solve $M_{IJ} = 0$ for all $I \in \mathcal{I}_{n,i}$ and $J \in \mathcal{I}_{n-1,i-1}$. The aim of this section is to reduce the number of cases for M_{IJ} to consider. We would like to emphasize that, consequently, no matter how large n is, it is sufficient to consider at most eleven cases. This is achieved in Proposition 6.19. As in Section 6.3, throughout this section, we assume that $n \geq 1$ and $1 \leq i \leq n$.

As Lemma 6.12 treats M_{IJ}^{scalar} for $M_{IJ} = M_{IJ}^{\text{scalar}} + M_{IJ}^{\text{vect}}$, it suffices to consider M_{IJ}^{vect} .

It follows from Proposition 4.9 that, for $\psi = \sum_{k=0}^2 (T_{a-k} g_k) h_{i \rightarrow i-1}^{(k)}$, we have

$$M_{IJ}^{\text{vect}} = \sum_{I' \in \mathcal{I}_{n,i}} A_{II'} \psi_{I'J} = \sum_{k=0}^2 \sum_{I' \in \mathcal{I}_{n,i}} A_{II'} \left(T_{a-k} g_k h_{I'J}^{(k)} \right),$$

where $A_{II'}$ is the vector field given in Lemma 5.3, namely,

$$(6.20) \quad A_{II'} = \begin{cases} \text{sgn}(I; \ell) \frac{\partial}{\partial \zeta_\ell} & \text{if } (I \setminus I') \amalg (I' \setminus I) = \{1, \ell\} \ (\ell \neq 1), \\ 0 & \text{otherwise.} \end{cases}$$

Then, in order to evaluate M_{IJ}^{vect} , one needs to compute $\sum_{I' \in \mathcal{I}_{n,i}} A_{II'} \psi_{I'J}$. However, in fact, almost all the terms $A_{II'} \psi_{I'J}$ are zero. We formulate it precisely by introducing the definition of $\text{Supp}(I, J; k)$ as follows.

Definition 6.13. For $I \in \mathcal{I}_{n,i}$, $J \in \mathcal{I}_{n-1,i-1}$, and $k \in \{0, 1, 2\}$, define a subset $\text{Supp}(I, J; k)$ of $\mathcal{I}_{n,i}$ by

$$\text{Supp}(I, J; k) := \{I' \in \mathcal{I}_{n,i} : A_{II'} \neq 0, \text{ and } h_{I'J}^{(k)} \neq 0\}.$$

It follow from (6.15), (6.16), and (6.17) that we have

$$\text{Supp}(I, J; k) \subset \begin{cases} \mathcal{I}_{n,i} \setminus \mathcal{I}_{n-1,i} & \text{for } k = 0, 2, \\ \mathcal{I}_{n-1,i} & \text{for } k = 1. \end{cases}$$

By using $\text{Supp}(I, J; k)$, M_{IJ}^{vect} may be given as follows:

$$(6.21) \quad M_{IJ}^{\text{vect}} = \sum_{k=0}^2 \left(\sum_{I' \in \text{Supp}(I, J; k)} A_{II'} \left(T_{a-k} g_k h_{I'J}^{(k)} \right) \right).$$

We now show that if $|J \setminus I|$ is large, then $\text{Supp}(I, J; k) = \emptyset$ and thus $M_{IJ}^{\text{vect}} = 0$. Together with the results in Lemma 6.12, this allows us to focus on the cases when $|J \setminus I|$ is small. In fact, it turns out that it suffices to consider only the cases when $|J \setminus I| \in \{0, 1\}$, see Lemma 6.15 below.

We first show that if $|J \setminus I| \geq 2$, then $M_{IJ}^{\text{vect}} = 0$. We prove it in two steps, namely, Lemmas 6.14 and 6.15. The following “triangle inequality” for arbitrary three sets I , I' , and J is used in the proof for Lemma 6.14:

$$(6.22) \quad |J \setminus I| \leq |J \setminus I'| + |I' \setminus I|.$$

Lemma 6.14. *We have the following:*

- (1) *If $|J \setminus I| \geq 2$, then $\text{Supp}(I, J; k) = \emptyset$ for $k = 0, 1$.*
- (2) *If $|J \setminus I| \geq 3$, then $\text{Supp}(I, J; 2) = \emptyset$.*

Consequently, if $|J \setminus I| \geq 3$, then $M_{IJ}^{\text{vect}} = 0$.

Proof. Observe that, for $k = 0, 1, 2$, if $\text{Supp}(I, J; k) \neq \emptyset$, then there exists I' so that $A_{II'} \neq 0$; in particular, $|I' \setminus I| = 1$ by (6.20). On the other hand, if $I' \in \text{Supp}(I, J; k)$, then $h_{I'J}^{(k)} \neq 0$, and therefore $|J \setminus I'| = 0$ for $k = 0, 1$ and $|J \setminus I'| \leq 1$ for $k = 2$ by (6.15)-(6.17). We then get $|J \setminus I| \leq 2$ for $k = 0, 1$ and $|J \setminus I| \leq 1$ for $k = 2$ by (6.22). \square

Lemma 6.15. *If $|J \setminus I| = 2$, then $M_{IJ}^{\text{vect}} = 0$.*

Proof. Under the condition $|J \setminus I| = 2$, we first observe $\text{Supp}(I, J; k) = \emptyset$ for $k = 0, 1$ by Lemma 6.14 (1). Further, for any $I' \in \text{Supp}(I, J; 2)$, $I' \ni n$ and $|J \setminus I'| \leq 1$ by (6.17). On the other hand, $|J \setminus I'| \geq |J \setminus I| - |I \setminus I'| = 2 - 1 = 1$ by (6.20). Hence $|J \setminus I'| = 1$.

Assume $n \notin I$. Then (I, I') must be of the form $I = K \cup \{1\}$ and $I' = K \cup \{n\}$ by (6.20), which is impossible because $2 = |J \setminus I| \leq |J \setminus K| = |J \setminus I'| = 1$. Hence $\text{Supp}(I, J; 2) = \emptyset$ if $n \notin I$ and $|J \setminus I| = 2$.

Assume now $n \in I$. Then there are two cases, namely, $1 \in I$ and $1 \notin I$, and in each case $\text{Supp}(I, J; 2)$ consists of two elements. Indeed, for $K := I \cap J \subset \mathcal{I}_{n-1, i-3}$, we have the following.

- (1) $I = K \cup \{1, r, n\}$, $J = K \cup \{p, q\}$ for some r :

$$\text{Supp}(I, J; 2) = \{K \cup \{p, r, n\}, K \cup \{q, r, n\}\}.$$

(2) $I = K \cup \{p, q, n\}$, $J = K \cup \{1, r\}$ for some r :

$$\text{Supp}(I, J; 2) = \{K \cup \{1, p, n\}, K \cup \{1, q, n\}\}.$$

In either case, we have $M_{IJ}^{\text{vect}} = 0$ if and only if $\zeta_r \left(\zeta_p \frac{\partial}{\partial \zeta_q} - \zeta_q \frac{\partial}{\partial \zeta_p} \right) (T_{a-2}g_2) = 0$, which clearly holds, as $T_{a-2}g_2$ is $O(n-1, \mathbb{C})$ -invariant. Now the assertion holds. \square

Remark 6.16. The case that $|J \setminus I| = 2$ happens only when $n \geq 5$ and $3 \leq i \leq n-2$.

Now we obtain the following lemma.

Lemma 6.17. *If $|J \setminus I| \geq 2$ then $M_{IJ}^{\text{vect}} = 0$.*

Proof. This follows from Lemmas 6.14 and 6.15. \square

By Lemmas 6.12 and 6.17, it suffices to focus on M_{IJ} with $|J \setminus I| \leq 1$. Observe that among the indices $1, 2, \dots, n$ for $\{1, 2, \dots, n\}$, “1” and “ n ” play special roles for I and J , as “1” comes from our choice of N_1^+ for $\widehat{d\pi_{(i,\lambda)}^*}(N_1^+)$ and as “ n ” makes the difference between $M = O(n) \times O(1)$ and $M' = O(n-1) \times O(1)$. On the other hand, all the pairs $(I, J) \in \mathcal{I}_{n,i} \times \mathcal{I}_{n-1,i-1}$ with $|J \setminus I| \leq 1$ are classified into $2^4 (= 16)$ cases according to whether each of the following conditions on (I, J) holds or not: $1 \in J$, $1 \in I$, $n \in I$, and $J \subset I$. For simplicity we represent them by quadruples $[\pm, \pm, \pm, \pm]$ as follows.

Definition 6.18. We mean by quadruples $[\pm, \pm, \pm, \pm]$ the cases according to whether each condition $1 \in J$, $1 \in I$, $n \in I$, and $J \subset I$ holds.

For instance, by $[-, +, -, +]$, we mean that (I, J) satisfies $1 \notin J$, $1 \in I$, $n \notin I$ and $J \subset I$.

Among $2^4 (= 16)$ cases for (I, J) with $|J \setminus I| \leq 1$, we show that at most eleven cases need to be taken into account, and thus Lemma 6.10 can be refined as follows.

Proposition 6.19. *Let $\psi = \sum_{k=0}^2 (T_{a-k}g_k)h_{i \rightarrow i-1}^{(k)}$. The the following two conditions on (g_0, g_1, g_2) are equivalent:*

- (i) $\widehat{d\pi_{(i,\lambda)}^*}(N_1^+)\psi = 0$.
- (ii) $M_{IJ} = 0$ for any $(I, J) \in \mathcal{I}_{n,i} \times \mathcal{I}_{n-1,i-1}$, subject to the eleven cases in Table 6.1:

TABLE 6.1. (I, J) for $1 \leq i \leq n$

	I	J	$[\pm, \pm, \pm, \pm]$
(1)	$J \cup \{n\}$	J	$++++$
(2)	$K \cup \{1, n\}$	$K \cup \{p\}$	$-++-$
(3)	$K \cup \{p, n\}$	$K \cup \{1\}$	$+--+$
(4)	$J \cup \{n\}$	J	$--++$
(5)	$K \cup \{p, n\}$	$K \cup \{q\}$	$--+-$
(6)	$K \cup \{p, q\}$	$K \cup \{1\}$	$+---$
(7)	$J \cup \{p\}$	J	$---+$
(8)	$J \cup \{p\}$	J	$++-+$
(9)	$K \cup \{1, p\}$	$K \cup \{q\}$	$-+--$
(10)	$J \cup \{1\}$	J	$-+-+$
(11)	$K \cup \{1, p, n\}$	$K \cup \{1, q\}$	$+++-$

For later convenience, we have described in Table 6.1 the general form of (I, J) for types (1)-(11), where each union is disjoint and $1, p, q, n$ are all distinct numbers. By this description, we observe that some of these types do not occur when i or $n - i$ is very small. To be precise, we have:

Remark 6.20. For $i = 1, n - 1$, or n , only the following cases occur:

- (a) $i = n$: (1);
- (b) $n = 2$ and $i = 1$: (4), (10);
- (c) $n \geq 3$:
 - (c1) $i = 1$: (4), (7), (10);
 - (c2) $i = n - 1$: (1), (2), (3), (4), (8), (10).

Hence Proposition 6.19 includes the following degenerate cases.

Proposition 6.21 ($i = 1$). *The following two conditions on (g_0, g_1) are equivalent:*

- (i) $\widehat{d\pi_{(1,\lambda)^*}(N_1^+)}\psi = 0$.
- (ii) $M_{IJ} = 0$ for any pair $(I, J) \in \mathcal{I}_{n,1} \times \mathcal{I}_{n-1,0}$ that belongs to (4), (7), (10) in Table 6.1, namely, for $(I, J) = (\{n\}, \emptyset), (\{p\}, \emptyset)$ ($1 \leq p \leq n - 1$), $(\{1\}, \emptyset)$.

Proposition 6.22 ($i = n$). *The following two conditions on g_0 are equivalent:*

- (i) $\widehat{d\pi_{(n,\lambda)^*}(N_1^+)}\psi = 0$.
- (ii) $M_{IJ} = 0$ for any $(I, J) \in \mathcal{I}_{n,n} \times \mathcal{I}_{n-1,n-1}$ that belongs to (1) in Table 6.1, namely, $(I, J) = (\{1, \dots, n\}, \{1, \dots, n - 1\})$.

The proof of Proposition 6.19 consists of several lemmas; nonetheless, it is basically done in two steps. First we observe that the three cases $[+, -, +, +]$, $[+, -, -, +]$,

and $[-, +, +, +]$ do not exist set-theoretically. We then show that, for the other two cases $[+, +, -, -]$ and $[-, -, -, -]$, we have $M_{IJ} = 0$.

Lemma 6.23. *Set-theoretically, the three cases $[+, -, +, +]$, $[+, -, -, +]$, and $[-, +, +, +]$ do not exist.*

Proof. If $1 \in J \subset I$, then $1 \in I$. Therefore $[+, -, \pm, +]$ do not exist. If $J \subset I$ and $n \in I$, then $J = I \setminus \{n\}$; in particular, in the case, if $1 \in I$, then $1 \in J$. Hence, $[-, +, +, +]$ does not exist. \square

Next we aim to show that $M_{IJ} = 0$ for (I, J) of type $[+, +, -, -]$ and $[-, -, -, -]$. More generally, we observe that the set $\text{Supp}(I, J; k)$ (see Definition 6.13) is determined by the types of $[\pm, \pm, \pm, \pm]$, and actually, this is the reason that we introduced the notation $[\pm, \pm, \pm, \pm]$ here. The simplest case is $k = 0$, where we have

$$\text{Supp}(I, J; 0) = \begin{cases} J \cup \{n\} & \text{for } [-, +, -, +] \text{ or } [+, -, +, -], \\ \emptyset & \text{otherwise.} \end{cases}$$

By using this idea we consider Lemmas 6.24 and 6.25 below.

Lemma 6.24. *The following hold.*

- (1) *If (I, J) is of type $[+, +, \pm, \pm]$, then $\text{Supp}(I, J; 1) = \emptyset$.*
- (2) *If (I, J) is of type $[+, +, -, -]$, then $\text{Supp}(I, J; k) = \emptyset$ for $k = 0, 1, 2$.*

Proof. For the first statement suppose that $1 \in I \cap J$. If $I' \in \mathcal{I}_{n,i}$ satisfies $A_{II'} \neq 0$, then $1 \notin I'$ as $1 \in I$. Hence $J \not\subset I'$ since $1 \in J$. Therefore $h_{I'J}^{(1)} = 0$.

To show the second statement, it suffices to show that $J \subset I$ if $1 \in J$, $1 \in I \not\subset n$, and $\text{Supp}(I, J; k) \neq \emptyset$ for $k = 0$ or 2 . For $k = 0, 2$, let $I' \in \text{Supp}(I, J; k)$. By (6.20), $I' = I \setminus \{1\} \cup \{n\}$ because $1 \in I \not\subset n$ and $n \in I'$. Then $|J \setminus (I' \setminus \{n\})| = 1 + |J \setminus I|$ because

$$J \setminus (I' \setminus \{n\}) = J \setminus (I \setminus \{1\}) = \{1\} \cup (J \setminus I).$$

Since $h_{I'J}^{(k)} \neq 0$ implies that $|J \setminus (I' \setminus \{n\})| \leq 1$, this shows that $J \subset I$. \square

Lemma 6.25. *The following hold.*

- (1) *If (I, J) is of type $[-, -, -, \pm]$, then $\text{Supp}(I, J; 0) = \text{Supp}(I, J; 2) = \emptyset$.*
- (2) *If (I, J) is of type $[-, -, -, -]$, then $\text{Supp}(I, J; k) = \emptyset$ for $k = 0, 1, 2$.*

Proof. For the first statement observe that if $A_{II'} \neq 0$, then $I' \subset I \cup \{1\}$ because $1 \notin I$. Since $n \notin I$, we have $n \notin I'$, which shows that $\text{Supp}(I, J; k) = \emptyset$ for $k = 0, 2$ by (6.18). To show the second statement, it suffices to show that $J \subset I$ if $1 \notin J$, $1 \notin I \not\subset n$, and $\text{Supp}(I, J; 1) \neq \emptyset$. Let $I' \in \text{Supp}(I, J; 1)$. Since $A_{II'} \neq 0$, we have $I' \subset I \cup \{1\}$ by (6.20) because $1 \notin I$. Since $h_{I'I}^{(1)} \neq 0$, we have $J \subset I'$ by (6.16). Thus $J \subset I \cup \{1\}$. Therefore $J \subset I$ as $1 \notin J$. \square

Lemma 6.26. *For the cases $[+, +, -, -]$ and $[-, -, -, -]$, we have $M_{IJ} = 0$.*

Proof. By Lemma 6.12, we have $M_{IJ}^{\text{scalar}} = 0$. Moreover, it follows from Lemmas 6.24 and 6.25 that $M_{IJ}^{\text{vect}} = 0$. As $M_{IJ} = M_{IJ}^{\text{scalar}} + M_{IJ}^{\text{vect}}$, this proves the lemma. \square

Proof for Proposition 6.19. The assertion follows from Lemmas 6.23 and 6.26. \square

6.5. Step 4 - Part I: Formulæ for saturated differential equations. The goal of Step 4 is to express the matrix coefficients M_{IJ} for (I, J) in Table 6.1 in terms of $L_r(g_0, g_1, g_2)$ for $r = 1, \dots, 7$. In this short section we collect several useful formulæ. The actual expressions for M_{IJ} are obtained in the next section.

Recall from (4.4) that we have defined a multi-valued meromorphic function $T_a g(\zeta)$ of n variables $\zeta = (\zeta_1, \dots, \zeta_n)$ by

$$(T_a g)(\zeta) = Q_{n-1}(\zeta')^{\frac{a}{2}} g\left(\frac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}\right)$$

for $a \in \mathbb{N}$ and $g(t) \in \mathbb{C}[t]$, where $Q_{n-1}(\zeta') = \zeta_1^2 + \dots + \zeta_{n-1}^2$. As in [21, Sect. 3.2], we say that a differential operator D on \mathbb{C}^n is *T-saturated* if there exists an operator E on $\mathbb{C}[t]$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}[t] & \xrightarrow{T_a} & \mathbb{C}(\zeta_1, \dots, \zeta_n) \\ E \downarrow & & \downarrow D \\ \mathbb{C}[t] & \xrightarrow{T_a} & \mathbb{C}(\zeta_1, \dots, \zeta_n). \end{array}$$

Such an operator E is unique as far as it exists. We denote the operator E by $T_a^\sharp D$. We allow D to have meromorphic coefficients. We note that

$$T_a^\sharp(D_1 \cdot D_2) = T_a^\sharp(D_1) \cdot T_a^\sharp(D_2)$$

whenever it makes sense. For more general definition of *T-saturated* differential operators see [21, Sect. 3.2].

Lemma 6.27. *Let R_ℓ^μ be the differential operator defined in (4.7) and $\vartheta_t = t \frac{d}{dt}$ be the Euler operator. For $a \in \mathbb{N}$ and $g(t) \in \text{Pol}_a[t]_{\text{even}}$ (see (4.5)), the following hold:*

- (1) $(T_a g)(\zeta) = Q_{n-1}(\zeta')(T_{a-2} g)(\zeta),$
- (2) $T_a(tg(t))(\zeta) = \zeta_n(T_{a-1} g)(\zeta),$
- (3) $\frac{\partial}{\partial \zeta_m}(T_a g)(\zeta) = \frac{\zeta_m}{Q_{n-1}(\zeta')} T_a((a - \vartheta_t)g)(\zeta) \quad (1 \leq m \leq n-1),$
- (4) $\frac{\partial}{\partial \zeta_n}(T_a g)(\zeta) = T_{a-1}\left(\frac{dg}{dt}\right)(\zeta),$
- (5) $T_\ell^\sharp\left(\frac{Q_{n-1}(\zeta')}{\zeta_m} \frac{\partial}{\partial \zeta_m}\right) = \ell - \vartheta_t \quad (1 \leq m \leq n-1),$
- (6) $T_\ell^\sharp\left(\frac{Q_{n-1}(\zeta')}{\zeta_m} \widehat{d\pi_{\lambda^*}}(N_m^+)\right) = R_\ell^{\lambda - \frac{n-1}{2}} \quad (1 \leq m \leq n-1).$

Proof. Formula (6) is a restatement of Lemma 4.7. Formula (5) is shown in [21, Lem. 6.10]. Also (1), \dots , (4) can be verified in the same spirit. \square

Lemma 6.28. *For $\mu \in \mathbb{C}$ and $\ell \in \mathbb{N}$, we have*

$$(6.23) \quad R_\ell^{\mu+1} = R_\ell^\mu + (\ell - \vartheta_t).$$

Proof. This immediately follows from the definition of R_ℓ^μ . \square

6.6. Step 4 - Part II: Explicit formulæ for M_{IJ} . In this section, by using the formulæ in Lemma 6.27, we express M_{IJ} for (I, J) in Table 6.1 in terms of $L_r \equiv L_r(g_0, g_1, g_2)$ for $r = 1, \dots, 7$. For $J \in \mathcal{I}_{n,i-1}$, we write $Q_J(\zeta') = \sum_{m \in J} \zeta_m^2$.

Lemma 6.29. *Let $n \geq 3$ and $1 \leq i \leq n$. For each case of (1), \dots , (11) in Table 6.1, M_{IJ} is given as follows:*

- (1) $M_{IJ} = (-1)^{i-1} \zeta_1 (Q_J(\zeta') T_{a-4}(L_1) + T_{a-2}(L_7 - \frac{i-1}{n-1} L_1 + \frac{n-i}{n-1} L_4)),$
- (2) $M_{IJ} = (-1)^{i-1} \text{sgn}(K; p) \zeta_p (\zeta_1^2 T_{a-4}(L_1) + T_{a-2}(L_6)),$
- (3) $M_{IJ} = (-1)^{i-1} \text{sgn}(K; p) \zeta_p (\zeta_1^2 T_{a-4}(L_1) + T_{a-2}(L_4 - L_6)),$
- (4) $M_{IJ} = (-1)^{i-1} \zeta_1 (Q_J(\zeta') T_{a-4}(L_1) + T_{a-2}(L_7 - \frac{i-1}{n-1} L_1 - \frac{i-1}{n-1} L_4)),$
- (5) $M_{IJ} = (-1)^{i-1} \text{sgn}(K; p, q) \zeta_1 \zeta_p \zeta_q T_{a-4}(L_1),$
- (6) $M_{IJ} = 0,$
- (7) $M_{IJ} = \text{sgn}(J; p) \zeta_1 \zeta_p T_{a-3}(L_2),$
- (8) $M_{IJ} = \text{sgn}(J; p) \zeta_1 \zeta_p T_{a-3}(L_2 - L_3),$
- (9) $M_{IJ} = \text{sgn}(K; p, q) \zeta_p \zeta_q T_{a-3}(L_3),$
- (10) $M_{IJ} = \zeta_1^2 T_{a-3}(L_2) + Q_J(\zeta') T_{a-3}(L_3) - T_{a-1}(L_3 + L_5),$
- (11) $M_{IJ} = (-1)^{i-1} \text{sgn}(K; p, q) \zeta_1 \zeta_p \zeta_q T_{a-4}(L_1).$

Remark 6.30. Suppose $i = 1$. Then, only (4), (7), and (10) occur (see Remark 6.20) on one hand, $g_2 = 0$ by (6.14) on the other hand. Therefore M_{IJ} in Lemma 6.29 amounts to

- (4) $M_{IJ} = \zeta_1(T_{a-2}L_7),$
- (5) $M_{IJ} = \zeta_1\zeta_p(T_{a-3}L_2),$
- (8) $M_{IJ} = \zeta_1^2(T_{a-3}L_2) - T_{a-1}(L_3 + L_5).$

Remark 6.31. Suppose $i = n$. Then only (1) occurs (see Remark 6.20) on one hand, $g_1 = g_2 = 0$ by (6.14) on the other hand. Therefore, M_{IJ} in Lemma 6.29 is given by $M_{IJ} = (-1)^{n-1}\zeta_1T_{a-2}(L_7).$

Proof. We only demonstrate two cases explicitly, namely, Cases (3) and (6); the other nine cases can be shown similarly. We choose Case (6) as an easy example and Case (3) as the most complicated example.

Case (6): $I = K \cup \{p, q\}$, $J = K \cup \{1\}$.

We wish to show that $M_{IJ} = 0$. Since $n \notin I$, by Lemma 6.11, M_{IJ}^{scalar} is given by

$$M_{IJ}^{\text{scalar}} = \frac{\zeta_1}{Q_{n-1}(\zeta')} T_{a-1} \left(R_{a-1}^{\lambda - \frac{n-1}{2}} g_1 \right) h_{IJ}^{(1)} + (\lambda + a - 1) T_{a-1} g_1 \frac{\partial h_{IJ}^{(1)}}{\partial \zeta_1}.$$

As $I \not\supset J$, we have $h_{IJ}^{(1)}(\zeta) = 0$ by (6.16). Therefore, $M_{IJ}^{\text{scalar}} = 0$. To evaluate M_{IJ}^{vect} , observe that we have

$$\begin{aligned} \text{Supp}(I, J; 0) &= \emptyset, \\ \text{Supp}(I, J; 1) &= \{K \cup \{1, p\}, K \cup \{1, q\}\}, \\ \text{Supp}(I, J; 2) &= \emptyset. \end{aligned}$$

It then follows from (6.21) and Lemma 6.27 (1) and (3) that

$$\begin{aligned} M_{IJ}^{\text{vect}} &= \sum_{k=0,2} \sum_{I' \in \text{Supp}(I, J; k)} A_{II'} \left(T_a g_0 h_{I'J}^{(k)} \right) + \sum_{I' \in \text{Supp}(I, J; 1)} A_{II'} \left(T_{a-1} g_1 h_{I'J}^{(1)} \right) \\ &= A_{K \cup \{p, q\}, K \cup \{1, p\}} \left(h_{K \cup \{1, p\}, K \cup \{1\}}^{(1)}(\zeta) T_{a-1} g_1 \right) + A_{K \cup \{p, q\}, K \cup \{1, q\}} \left(h_{K \cup \{1, q\}, K \cup \{1\}}^{(1)}(\zeta) T_{a-1} g_1 \right) \\ &= -(\text{sgn}(K \cup \{p\}; q) \text{sgn}(K; p) + \text{sgn}(K \cup \{q\}; p) \text{sgn}(K; q)) \zeta_p \zeta_q T_{a-3} ((a-1 - \vartheta_t) g_1), \end{aligned}$$

which vanishes by Lemma 5.2 (4). Hence we obtain $M_{IJ} = M_{IJ}^{\text{scalar}} + M_{IJ}^{\text{vect}} = 0$.

Case (3): $I = K \cup \{p, n\}$, $J = K \cup \{1\}$.

We wish to show that

$$(6.24) \quad M_{IJ} = (-1)^{i-1} \operatorname{sgn}(K; p) \zeta_p \left(\zeta_1^2 T_{a-4} \left(R_{a-2}^{\lambda - \frac{n-3}{2}} g_2 \right) \right. \\ \left. + T_{a-2} \left((a - \vartheta_t) g_0 - \frac{dg_1}{dt} + \left(\lambda + a - i + 1 - \frac{i-1}{n-1} (a - \vartheta_t) \right) g_2 \right) \right).$$

As in Case (6), we evaluate M_{IJ}^{scalar} and M_{IJ}^{vect} , separately. To begin with, we claim that

$$(6.25) \quad M_{IJ}^{\text{scalar}} = (-1)^{i-1} \operatorname{sgn}(K; p) \zeta_p \left(\zeta_1^2 T_{a-4} \left(R_{a-2}^{\lambda - \frac{n-1}{2}} g_2 \right) + T_{a-2}((\lambda + a - 1) g_2) \right).$$

First observe that, as $I \neq J$, we have $h_{IJ}^{(0)}(\zeta) = 0$. Then, by Lemma 6.11, M_{IJ}^{scalar} is given by

$$M_{IJ}^{\text{scalar}} = \frac{\zeta_1}{Q_{n-1}(\zeta')} T_{a-2} \left(R_{a-2}^{\lambda - \frac{n-1}{2}} g_2 \right) h_{IJ}^{(2)} + (\lambda + a - 1) T_{a-2} g_2 \frac{\partial h_{IJ}^{(2)}}{\partial \zeta_1} \\ =: (S1) + (S2),$$

as $n \in I$. By (1) of Lemma 6.27, we have

$$\frac{\zeta_1}{Q_{n-1}(\zeta')} T_{a-2} \left(R_{a-2}^{\lambda - \frac{n-1}{2}} g_2 \right) = \zeta_1 T_{a-4} \left(R_{a-2}^{\lambda - \frac{n-1}{2}} g_2 \right).$$

Moreover, $h_{IJ}^{(2)}(\zeta)$ is given by

$$h_{IJ}^{(2)}(\zeta) = h_{K \cup \{p, n\}, K \cup \{1\}}^{(2)}(\zeta) = (-1)^{i-1} \operatorname{sgn}(K \cup \{p, n\}; p) \zeta_1 \zeta_p = (-1)^{i-1} \operatorname{sgn}(K; p) \zeta_1 \zeta_p.$$

Therefore,

$$(S1) = (-1)^{i-1} \operatorname{sgn}(K; p) \zeta_p \zeta_1^2 T_{a-4} \left(R_{a-2}^{\lambda - \frac{n-1}{2}} g_2 \right) \text{ and } (S2) = (-1)^{i-1} \operatorname{sgn}(K; p) \zeta_p T_{a-2}((\lambda + a - 1) g_2).$$

Now (6.25) follows from (S1) and (S2).

To evaluate M_{IJ}^{vect} , observe that we have

$$(6.26) \quad \begin{aligned} \operatorname{Supp}(I, J; 0) &= \{K \cup \{1, n\}\}, \\ \operatorname{Supp}(I, J; 1) &= \{K \cup \{1, p\}\}, \\ \operatorname{Supp}(I, J; 2) &= \{K \cup \{1, n\}\} \cup \bigcup_{r \in K} \{(K \setminus \{r\}) \cup \{1, p, n\}\}. \end{aligned}$$

Accordingly, we decompose M_{IJ}^{vect} as $M_{IJ}^{\text{vect}} = (M0) + (M1) + (M2)$, where we set

$$(Mk) := \sum_{I' \in \operatorname{Supp}(I, J; k)} A_{II'} \left(T_{a-k} g_k h_{I'J}^{(k)} \right), \\ (M'k) := (-1)^{i-1} \operatorname{sgn}(K; p) \zeta_p^{-1} (Mk)$$

for $k = 0, 1, 2$. We claim that

$$(6.27) \quad (M'0) = T_{a-2}((a - \vartheta_t)g_0),$$

$$(6.28) \quad (M'1) = -T_{a-2} \left(\frac{dg_1}{dt} \right),$$

$$(6.29) \quad (M'2) = T_{a-4}((a - 2 - \vartheta_t)g_2) - T_{a-2} \left((i - 2) + \frac{i - 1}{n - 1} (a - \vartheta_t)g_2 \right).$$

Indeed, for $(M0)$, we have

$$\begin{aligned} (M0) &= \sum_{I' \in \text{Supp}(I, J; 0)} A_{II'} \left(T_a g_0 h_{I'J}^{(0)} \right) \\ &= A_{K \cup \{p, n\}, K \cup \{1, n\}} \left(h_{K \cup \{1, n\}, K \cup \{1\}}^{(0)}(\zeta) T_a g_0 \right) \\ &= (-1)^{i-1} \text{sgn}(K; p) \frac{\partial}{\partial \zeta_p} (T_a g_0). \end{aligned}$$

Now (6.27) follows from (1) and (3) of Lemma 6.27. (6.28) can be shown similarly.

Then, we have from (6.26)

$$\begin{aligned} (M2) &= \sum_{I' \in \text{Supp}(I, J; 2)} A_{II'} \left(T_{a-2} g_2 h_{I'J}^{(2)} \right) \\ &= A_{K \cup \{p, n\}, K \cup \{1, n\}} \left(h_{K \cup \{1, n\}, K \cup \{1\}}^{(2)} T_{a-2} g_2 \right) \\ &\quad + \sum_{r \in K} A_{K \cup \{p, n\}, (K \setminus \{r\}) \cup \{1, p, n\}} \left(h_{(K \setminus \{r\}) \cup \{1, p, n\}, K \cup \{1\}}^{(2)}(\zeta) T_{a-2} g_2 \right). \end{aligned}$$

By the formula of $h_{I'J}^{(2)}$ in Table 5.1 and a computation of signature

$$(6.30) \quad \text{sgn}(K \cup \{p\}; r) \text{sgn}(K \cup \{p\}; p, r) = -\text{sgn}(K \cup \{p\}; p) = -\text{sgn}(K; p)$$

from Lemma 5.2 (1) and (3), we have

$$(6.31) \quad (M'2) = \zeta_p^{-1} \frac{\partial}{\partial \zeta_p} \left(\tilde{Q}_{K \cup \{1\}}(\zeta') T_{a-2} g_2 \right) - \sum_{r \in K} \frac{\partial}{\partial \zeta_r} (\zeta_r T_{a-2} g_2),$$

where $\tilde{Q}_{K \cup \{1\}}(\zeta') = Q_{K \cup \{1\}}(\zeta') - \frac{i-1}{n-1} Q_{n-1}(\zeta')$. By applying the formulæ in Lemma 6.27 accordingly, (6.31) is evaluated to

$$\begin{aligned}
(6.31) &= \zeta_p^{-1} \frac{\partial}{\partial \zeta_p} \left(\tilde{Q}_{K \cup \{1\}}(\zeta') T_{a-2} g_2 \right) - \sum_{r \in K} \frac{\partial}{\partial \zeta_r} (\zeta_r T_{a-2} g_2) \\
&= \zeta_p^{-1} \frac{\partial}{\partial \zeta_p} \left(Q_{K \cup \{1\}}(\zeta') T_{a-2} g_2 - \frac{i-1}{n-1} T_a g_2 \right) - \sum_{r \in K} (T_{a-2} g_2 + \zeta_r^2 T_{a-4}((a-2-\vartheta_t)g_2)) \\
&= Q_{K \cup \{1\}}(\zeta') T_{a-4}((a-2-\vartheta_t)g_2 - \frac{i-1}{n-1} T_{a-2}((a-\vartheta_t)g_2) \\
&\quad - ((i-2)T_{a-2}g_2 + (Q_K(\zeta')T_{a-4}((a-2-\vartheta_t)g_2))) \\
&= \zeta_1^2 T_{a-4}((a-2-\vartheta_t)g_2) - T_{a-2} \left((i-2) + \frac{i-1}{n-1} (a-\vartheta_t)g_2 \right).
\end{aligned}$$

Thus, (6.29) holds.

Now, by using Lemma 6.28, one obtains (6.24) from (6.25), (6.27), (6.28), and (6.29) as $M_{IJ} = M_{IJ}^{\text{scalar}} + (M0) + (M1) + (M2)$. This completes the proof for Case (3). \square

6.7. Step 5: Deduction from $M_{IJ} = 0$ to $L_r(g_0, g_1, g_2) = 0$. In this final step we deduce $L_r(g_0, g_1, g_2) = 0$ from $M_{IJ} = 0$. The following observation is useful.

Lemma 6.32. *Let p_1, p_2 be $O(n-1, \mathbb{C})$ -invariant polynomials in $\text{Pol}(\mathbb{C}^n)$ and $R \subsetneq \{1, \dots, n-1\}$. If $(\sum_{r \in R} \zeta_r^2) p_1 + p_2 = 0$, then $p_1 = p_2 = 0$.*

Proof. If $p_1 \neq 0$, then it follows from the hypothesis that $\sum_{r \in R} \zeta_r^2 = \frac{-p_2}{p_1}$ is $O(n-1, \mathbb{C})$ -invariant. However, since $R \subsetneq \{1, \dots, n-1\}$, we have $\sum_{r \in R} \zeta_r^2 \notin \text{Pol}(\mathbb{C}^n)^{O(n-1, \mathbb{C})}$. Hence, $p_1 = 0$ and, consequently, $p_2 = 0$. \square

Proposition 6.33. *Let $n \geq 3$ and $1 \leq i \leq n$. Let $g_k \in \text{Pol}_{a-k}[t]_{\text{even}}$ ($k = 0, 1, 2$).*

- (1) *Suppose $i = 1$. The following two conditions on (g_0, g_1) are equivalent:*
 - (i) $M_{IJ} = 0$ for all $I \in \mathcal{I}_{n,i}$ and $J \in \mathcal{I}_{n-1,i-1}$.
 - (ii) $L_r(g_0, g_1, 0) = 0$ ($r = 2, 7, 9$).
- (2) *Suppose $2 \leq i \leq n-1$. The following two conditions on (g_0, g_1, g_2) are equivalent:*
 - (i) $M_{IJ} = 0$ for all $I \in \mathcal{I}_{n,i}$ and $J \in \mathcal{I}_{n-1,i-1}$.
 - (ii) $L_r(g_0, g_1, g_2) = 0$ for all $r = 1, \dots, 7$.
- (3) *Suppose $i = n$. The following two conditions on (g_0, g_1, g_2) are equivalent:*
 - (i) $M_{IJ} = 0$ for all $I \in \mathcal{I}_{n,i}$ and $J \in \mathcal{I}_{n-1,i-1}$.
 - (ii) $L_7(g_0, g_1, g_2) = 0$.

Proof. (1) Suppose $i = 1$. Then the equivalence follows from Proposition 6.21 and Remark 6.30.

(2) Suppose $2 \leq i \leq n - 2$. By Proposition 6.19, we can replace condition (i) with the condition that $M_{IJ} = 0$ for all (I, J) in (1), \dots , (11) in Table 6.1. By Lemma 6.29, the implication from (ii) to (i) is then clear. The other implication also easily follows from Lemmas 6.29 and 6.32, as $T_b(g(t))$ is $O(n - 1, \mathbb{C})$ -invariant for any $b \in \mathbb{N}$ (see (4.6)). For $i = n - 1$, we can replace condition (i) with the condition that $M_{IJ} = 0$ for the six cases (1), \dots , (4), (8), (10) in Table 6.1, as we saw in (c2) of Remark 6.20. If $M_{IJ} = 0$ for the six cases, we still get $L_r(g_0, g_1, g_2) = 0$ for $r = 1, 2, \dots, 7$ by Lemmas 6.29 and 6.32. Thus the implication (i) \Rightarrow (ii) is verified also for $i = n - 1$. The converse implication is clear.

(3) Suppose $i = n$. The equivalence follows from Proposition 6.22 and Remark 6.31 with the same argument as above. \square

Now we give a proof for Theorem 6.5, as a summary of this section.

Proof for Theorem 6.5. The equivalence of the statements follow from Lemma 3.4, Lemma 6.10, and Propositions 6.19 and 6.33. \square

7. F-SYSTEM FOR SYMMETRY BREAKING OPERATORS ($j = i - 2, i + 1$ CASE)

In this chapter we solve the F-system for $j = i + 1$, and give a complete classification of differential symmetry breaking operators which raise the degree of differential forms by one or decrease the degree by two,

$$I(i, \lambda)_\alpha \longrightarrow J(i + 1, \nu)_\beta,$$

$$I(i, \lambda)_\alpha \longrightarrow J(i - 2, \nu)_\beta,$$

for $\lambda, \nu \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$.

In contrast to the case with $j = i - 1, i$ that was treated in Chapter 6, we see that there are not many differential symmetry breaking operators for $j = i - 2$ or $i + 1$. Here are the main results of this chapter, which are a part of Theorem 1.1 ($j = i - 2, i + 1$ case):

Theorem 7.1. *Suppose $0 \leq i \leq n - 2$, $\lambda, \nu \in \mathbb{C}$, and $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$. Then the following three conditions on $(i, \lambda, \nu, \alpha, \beta)$ are equivalent:*

- (i) $\text{Diff}_{G'}(I(i, \lambda)_\alpha, J(i + 1, \nu)_\beta) \neq \{0\}$.
- (ii) $\dim_{\mathbb{C}} \text{Diff}_{G'}(I(i, \lambda)_\alpha, J(i + 1, \nu)_\beta) = 1$.
- (iii) $\lambda \in \{0, -1, -2, \dots\}$, $\nu = 1$, $\beta \equiv \alpha + \lambda + 1 \pmod{2}$ when $i = 0$;
 $\lambda = i$, $\nu = i + 1$, $\beta \equiv \alpha + 1 \pmod{2}$ when $i \geq 1$.

Theorem 7.2. *Suppose $2 \leq i \leq n$, $\lambda, \nu \in \mathbb{C}$, and $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$. Then the following three conditions on $(i, \lambda, \nu, \alpha, \beta)$ are equivalent:*

- (i) $\text{Diff}_{G'}(I(i, \lambda)_\alpha, J(i - 2, \nu)_\beta) \neq \{0\}$.
- (ii) $\dim_{\mathbb{C}} \text{Diff}_{G'}(I(i, \lambda)_\alpha, J(i - 2, \nu)_\beta) = 1$.
- (iii) $\lambda \in \{0, -1, -2, \dots\}$, $\nu = 1$, $\beta \equiv \alpha + \lambda + 1 \pmod{2}$ when $i = n$;
 $(\lambda, \nu) = (n - i, n - i + 1)$, $\beta \equiv \alpha + 1 \pmod{2}$ when $2 \leq i \leq n - 1$.

For the proof of Theorems 7.1 and 7.2, we first observe that the latter is derived from the former. In fact, the duality theorem for symmetry breaking operators (see Theorem 2.7) implies that there is a natural bijection:

$$\text{Diff}_{G'}(I(i, \lambda)_\alpha, J(i - 2, \nu)_\beta) \simeq \text{Diff}_{G'}(I(\tilde{i}, \lambda)_\alpha, J(\tilde{i} + 1, \nu)_\beta)$$

where $\tilde{i} := n - i$. Then it is easy to see that $(\tilde{i}, \lambda, \nu, \alpha, \beta)$ satisfies the condition (iii) in Theorem 7.1 if and only if $(i, \lambda, \nu, \alpha, \beta)$ satisfies the condition (iii) in Theorem 7.2, whence we conclude that Theorem 7.2 follows from Theorem 7.1 applied to the right-hand side. The rest of this chapter is devoted to the proof of Theorem 7.1.

7.1. Proof of Theorem 7.1. We have seen in (5.31) that the F-method gives a natural isomorphism

$$\text{Diff}_{G'}(I(i, \lambda)_\alpha, J(i + 1, \nu)_\beta) \simeq \text{Sol}\left(\mathfrak{n}_+; \sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(i+1)}\right),$$

where $Sol(\mathbf{n}_+; \sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(i+1)})$ is the space of $\text{Hom}_{\mathbb{C}}(\bigwedge^i(\mathbb{C}^n), \bigwedge^{i+1}(\mathbb{C}^{n-1}))$ -valued polynomial solutions on $\mathbf{n}_+ \simeq \mathbb{C}^n$ to the F-system associated to the outer tensor product representations $\sigma_{\lambda, \alpha}^{(i)} = \bigwedge^i(\mathbb{C}^n) \boxtimes (-1)^\alpha \boxtimes \mathbb{C}_\lambda$ and $\tau_{\nu, \beta}^{(i+1)} = \bigwedge^{i+1}(\mathbb{C}^{n-1}) \boxtimes (-1)^\beta \boxtimes \mathbb{C}_\nu$, of L and L' , respectively. Then Theorem 7.1 is deduced from the following explicit results.

Theorem 7.3. *Suppose $0 \leq i \leq n-2$. We recall from (5.27) that $h_{i \rightarrow i+1}^{(1)}: \bigwedge^i(\mathbb{C}^n) \rightarrow \bigwedge^{i+1}(\mathbb{C}^{n-1}) \otimes \mathcal{H}^1(\mathbb{C}^{n-1})$ is a nonzero $O(n-1)$ -homomorphism. Let $\lambda, \nu \in \mathbb{C}$, and $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$. Then*

$$\begin{aligned} & Sol\left(\mathbf{n}_+; \sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(i+1)}\right) \\ = & \begin{cases} \mathbb{C}\left(T_{-\lambda} \tilde{C}_{-\lambda}^{\lambda - \frac{n-1}{2}}\left(e^{\frac{\pi\sqrt{-1}}{2}} t\right)\right) h_{i \rightarrow i+1}^{(1)} & \text{if } \nu = 1, -\lambda \in \mathbb{N}, \beta - \alpha \equiv 1 - \lambda \pmod{2}, i = 0, \\ \mathbb{C} \cdot h_{i \rightarrow i+1}^{(1)} & \text{if } (\lambda, \nu) = (i, i+1), \beta \equiv \alpha + 1 \pmod{2}, 1 \leq i \leq n-2, \\ \{0\} & \text{otherwise.} \end{cases} \end{aligned}$$

In order to determine $Sol(\mathbf{n}_+; \sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(i+1)})$, we begin with a description of $\text{Hom}_{L'}(\sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(i+1)}) \otimes \text{Pol}[\zeta_1, \dots, \zeta_n]$.

Lemma 7.4. *Suppose that $0 \leq i \leq n-2$. Then,*

$$\begin{aligned} & \text{Hom}_{L'}\left(\sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(i+1)} \otimes \text{Pol}[\zeta_1, \dots, \zeta_n]\right) \\ \simeq & \begin{cases} \left\{ (T_{\nu-\lambda-1} g) h_{i \rightarrow i+1}^{(1)} : g \in \text{Pol}_{\nu-\lambda-1}[t]_{\text{even}} \right\} & \text{if } \nu - \lambda \in \mathbb{N}_+ \text{ and } \beta - \alpha \equiv \nu - \lambda \pmod{2}, \\ \{0\} & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. The statement follows from Proposition 5.17 and Lemma 5.18. \square

From now, assume $\nu - \lambda \in \mathbb{N}_+$ and $\beta - \alpha \equiv \nu - \lambda \pmod{2}$. We set

$$a := \nu - \lambda.$$

Then it follows from Proposition 4.1 and Lemma 7.4 that we have a bijection:

$$\left\{ g \in \text{Pol}_{a-1}[t]_{\text{even}} : \widehat{d\pi_{(i, \lambda)}^*}(N_1^+)(T_{a-1}g)h_{i \rightarrow i+1}^{(1)} = 0 \right\} \xrightarrow{\sim} Sol\left(\mathbf{n}_+; \sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(i+1)}\right),$$

by $g \mapsto \psi := (T_{a-1}g)h_{i \rightarrow i+1}^{(1)}$.

Given $g \in \text{Pol}_{a-1}[t]_{\text{even}}$, we define ψ as above, and polynomials $M_{I\tilde{I}}$ of n variables ζ_1, \dots, ζ_n for $I \in \mathcal{I}_{n, i}$ and $\tilde{I} \in \mathcal{I}_{n-1, i+1}$ by

$$M_{I\tilde{I}} \equiv M_{I\tilde{I}}(g) := \langle \widehat{d\pi_{(i, \lambda)}^*}(N_1^+) \psi(e_I), e_{\tilde{I}}^\vee \rangle.$$

As in Section 4.5, clearly $\widehat{d\pi_{(i, \lambda)}^*}(N_1^+) \psi = 0$ if and only if $M_{I\tilde{I}} = 0$ for all $I \in \mathcal{I}_{n, i}$ and $\tilde{I} \in \mathcal{I}_{n-1, i+1}$.

Now the proof of Theorem 7.3 is reduced to the following lemma:

Lemma 7.5. *Suppose $a := \nu - \lambda \in \mathbb{N}_+$, $\beta \equiv \alpha + 1 \pmod{2}$, and $g(t) \in \text{Pol}_{a-1}[t]_{\text{even}}$ is a nonzero polynomial such that $M_{I\tilde{I}}(g) = 0$ for all $I \in \mathcal{I}_{n,i}$ and $\tilde{I} \in \mathcal{I}_{n-1,i+1}$.*

- (1) *If $i = 0$, then $\lambda = 1 - a$, $\nu = 1$, and g is proportional to $\tilde{C}_{a-1}^{\lambda - \frac{n-1}{2}} \left(e^{\frac{\pi\sqrt{-1}}{2}} t \right)$.*
- (2) *If $i \geq 1$, then $\lambda = i$, $\nu = i + 1$, $a = 1$ and $g(t)$ is a constant.*

In order to prove Lemma 7.5, we examine the matrix components $M_{I\tilde{I}}$ by decomposing

$$M_{I\tilde{I}} = M_{I\tilde{I}}^{\text{scalar}} + M_{I\tilde{I}}^{\text{vect}}$$

as in Proposition 4.9, corresponding to the decomposition of $\widehat{d\pi_{(i,\lambda)^*}(N_1^+)}$ into the scalar and vector parts. We use the following lemma.

Lemma 7.6. *For $I \in \mathcal{I}_{n,i}$ and $\tilde{I} \in \mathcal{I}_{n-1,i+1}$, we set*

$$\psi_{I\tilde{I}} := \langle \psi(e_I), e_{\tilde{I}}^{\vee} \rangle = (T_{a-1}g) \langle h_{i \rightarrow i+1}^{(1)}(e_I), e_{\tilde{I}}^{\vee} \rangle.$$

- (1) *We have*

$$\psi_{I\tilde{I}} = \begin{cases} \text{sgn}(I; p)(T_{a-1}g)\zeta_p & \text{if } \tilde{I} = I \cup \{p\}, \\ 0 & \text{if } \tilde{I} \not\supset I. \end{cases}$$

- (2) *$M_{I\tilde{I}}^{\text{scalar}} = 0$ if $\tilde{I} \not\supset I$. If $\tilde{I} = I \cup \{p\}$, then*

$$M_{I\tilde{I}}^{\text{scalar}} = \text{sgn}(I; p) \left(\frac{\zeta_1 \zeta_p}{Q_{n-1}(\zeta')} T_{a-1}(R_{a-1}^{\lambda - \frac{n-1}{2}} g) + (\lambda + a - 1) \delta_{p1} T_{a-1}g \right),$$

where δ_{p1} is the Kronecker delta.

- (3) *The vector part $M_{I\tilde{I}}^{\text{vect}}$ is given by*

$$M_{I\tilde{I}}^{\text{vect}} = \begin{cases} \sum_{q \in I} \text{sgn}(I; q) \frac{\partial}{\partial \zeta_q} \psi_{I \setminus \{q\} \cup \{1\}, \tilde{I}} & \text{if } 1 \notin I, \\ \sum_{q \notin I} \text{sgn}(I; q) \frac{\partial}{\partial \zeta_q} \psi_{I \setminus \{1\} \cup \{q\}, \tilde{I}} & \text{if } 1 \in I. \end{cases}$$

Proof. The first statement is immediate from Table 5.1 on the matrix coefficients of $h_{i \rightarrow j}^{(k)}$. The second statement follows from Proposition 4.4 (1), and the third one from Lemma 5.3 and Proposition 4.9. \square

We are ready to prove Lemma 7.5.

Proof of Lemma 7.5. (1) Suppose $i = 0$. Then $M_{I\tilde{I}}^{\text{vect}} = 0$ by Proposition 3.5. We note that $I = \emptyset$. Let $\tilde{I} = \{p\}$ ($1 \leq p \leq n - 1$). By Lemma 7.6 (3),

$$M_{I\tilde{I}} = M_{I\tilde{I}}^{\text{scalar}} = \frac{\zeta_1 \zeta_p}{Q_{n-1}(\zeta')} T_{a-1}(R_{a-1}^{\lambda - \frac{n-1}{2}} g) + \delta_{p1} (\lambda + a - 1) (T_{a-1}g).$$

Hence $M_{I\tilde{I}} = 0$ for all $\tilde{I} = \{p\}$ if and only if

$$R_{a-1}^{\lambda - \frac{n-1}{2}} g = 0 \quad \text{and} \quad \lambda + a - 1 = 0.$$

Thus the first assertion is obtained by Lemma 14.3 about the polynomial solutions to imaginary Gegenbauer differential equation $R_{a-1}^{\lambda - \frac{n-1}{2}} g = 0$.

(2) Let $i \geq 1$. Obviously $M_{I\tilde{I}} = 0$ for all I and \tilde{I} if g is a constant function. In order to prove the converse statement, we choose the following four cases.

Case 1. $I \subset \tilde{I}$ and $1 \notin \tilde{I}$.

Case 2. $I \subset \tilde{I}$ and $1 \in I$.

Case 3. $I \subset \tilde{I}$ and $1 \notin I$, $1 \in \tilde{I}$.

Case 4. $|I \setminus \tilde{I}| = 1$, $1 \notin I$, and $1 \in \tilde{I}$.

First we treat Case 1. We may write $\tilde{I} = I \cup \{p\}$. By Lemma 7.6, we have

$$\begin{aligned} M_{I\tilde{I}}^{\text{scalar}} &= \text{sgn}(I; p) \frac{\zeta_1 \zeta_p}{Q_{n-1}(\zeta')} T_{a-1}(R_a^{\lambda - \frac{n-1}{2}} g), \\ M_{I\tilde{I}}^{\text{vect}} &= 0. \end{aligned}$$

Hence the condition $M_{I\tilde{I}} = 0$ implies

$$(7.1) \quad R_{a-1}^{\lambda - \frac{n-1}{2}} g = 0.$$

Second, we treat Case 2. We may write $\tilde{I} = I \cup \{p\}$ with $p \neq 1$ and $1 \in I$. By using (7.1), we have $M_{I\tilde{I}}^{\text{scalar}} = 0$, whereas Lemma 7.6 (3) shows

$$M_{I\tilde{I}}^{\text{vect}} = \text{sgn}(I; p) \zeta_1 \frac{\partial}{\partial \zeta_p} (T_{a-1} g).$$

Hence the condition $M_{I\tilde{I}} = 0$ implies

$$(7.2) \quad \frac{\partial}{\partial \zeta_p} (T_{a-1} g) = 0.$$

By Lemma 6.27 (3), (7.2) yields an ordinary differential equation on $g(t)$:

$$(7.3) \quad (a - 1 - \vartheta_t) g(t) = 0,$$

where $\vartheta_t = t \frac{d}{dt}$. Third, we treat Case 4 before Case 3. We may write $I = K \cup \{n\}$ and $\tilde{I} = K \cup \{1, p\}$ with $K \in \mathcal{I}_{n-1, i-1}$ and $p \in \{2, \dots, n-1\} \setminus K$. Then, again by Lemma 7.6, $M_{I\tilde{I}}^{\text{scalar}} = 0$ and

$$\begin{aligned} M_{I\tilde{I}}^{\text{vect}} &= \text{sgn}(I; n) \frac{\partial}{\partial \zeta_n} \psi_{I \setminus \{n\} \cup \{1\}, \tilde{I}} \\ &= -\text{sgn}(K; p, n) \zeta_p \frac{\partial}{\partial \zeta_n} (T_{a-1} g). \end{aligned}$$

Hence the condition $M_{I\tilde{I}} = 0$ implies $\frac{\partial}{\partial \zeta_n}(T_{a-1}g) = 0$, and therefore we get

$$T_{a-2} \left(\frac{dg}{dt} \right) = 0$$

by Lemma 6.27 (4). Hence $g(t)$ is a constant. In turn, $a = 1$ by (7.3).

Finally, we consider Case 3, namely, $\tilde{I} = I \cup \{1\}$. Then

$$\begin{aligned} M_{I\tilde{I}}^{\text{scalar}} &= (\lambda + a - 1)T_{a-1}g, \\ M_{I\tilde{I}}^{\text{vect}} &= - \sum_{q \in I} \frac{\partial}{\partial \zeta_q}(T_{a-1}g)\zeta_q = -i(T_{a-1}g), \end{aligned}$$

where we have used (7.2) for $p \in \{2, \dots, n-1\}$. Hence we get

$$M_{I\tilde{I}} = (\lambda + a - 1 - i)T_{a-1}g,$$

and conclude $\lambda = i$. Hence the proof of Lemma 7.5 is completed. □

Thus we have proved Theorem 7.3, whence Theorem 7.1.

8. BASIC OPERATORS IN DIFFERENTIAL GEOMETRY AND CONFORMAL COVARIANCE

In this chapter we collect some elementary properties of basic operators such as the Hodge star operators, the codifferential d^* , and the interior multiplication $\iota_{N_Y(X)}$ by the normal vector field for hypersurfaces Y in pseudo-Riemannian manifolds X . These operators are obviously invariant under isometries, but also satisfy certain conformal covariance which we formulate in terms of the representations $\varpi_{u,\delta}^{(i)}$ ($u \in \mathbb{C}, \delta \in \mathbb{Z}/2\mathbb{Z}$), see (1.1), of the conformal group on the space $\mathcal{E}^i(X)$ of i -forms.

The conformal covariance of the Hodge star plays an important role in the classification of differential symmetry breaking operators as we have seen in Theorem 1.1 and shall see in Section 10.3, whereas that of the other operators such as d, d^* or $\iota_{N_Y(X)}$ is only a small part of the global conformal covariance of our symmetry breaking operators $\mathcal{D}_{u,a}^{i \rightarrow j}$.

8.1. Twisted pull-back of differential forms by conformal transformations.

Suppose (X, g_X) and $(X', g_{X'})$ are pseudo-Riemannian manifolds of the same dimension n . A local diffeomorphism $\Phi: X \rightarrow X'$ is said to be *conformal* if there exists a positive-valued function $\Omega \equiv \Omega_\Phi$ (*conformal factor*) on X such that

$$\Phi^*(g_{X', \Phi(x)}) = \Omega(x)^2 g_{X,x} \quad \text{for all } x \in X.$$

We define a locally constant function $\sigma(\Phi)$ on X by

$$\sigma(\Phi)(x) \equiv \sigma_X(\Phi)(x) = \begin{cases} 1 & \text{if } \Phi_{*x}: T_x X \longrightarrow T_{\Phi(x)} X \text{ is orientation-preserving,} \\ -1 & \text{if } \Phi_{*x}: T_x X \longrightarrow T_{\Phi(x)} X \text{ is orientation-reversing.} \end{cases}$$

The twisted pull-back $\Phi_{u,\delta}^* \equiv \left(\Phi_{u,\delta}^{(i)} \right)^*$ with parameters $u \in \mathbb{C}$ and $\delta \in \mathbb{Z}/2\mathbb{Z}$ on i -forms is defined by

$$(8.2) \quad \Phi_{u,\delta}^*: \mathcal{E}^i(X') \longrightarrow \mathcal{E}^i(X), \quad \alpha \mapsto \sigma(\Phi)^\delta \Omega^u \Phi^* \alpha.$$

If $X = X'$ and G is the conformal group of X acting by $x \mapsto L_h x$ ($h \in G$), then the representation $\varpi_{u,\delta}^{(i)}$ of G on $\mathcal{E}^i(X)$ introduced in (1.1) is written as

$$(8.3) \quad \varpi_{u,\delta}^{(i)}(h) = \left((L_{h^{-1}})_{u,\delta}^{(i)} \right)^*.$$

8.2. Hodge star operators under conformal transformations.

We recall the standard notion of the Hodge star operator, and fix some notations. Given an oriented real vector space V of dimension $n = p + q$ equipped with a

nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ of signature (p, q) , we have canonical isomorphisms $V \simeq V^\vee$ and $\bigwedge^n V \simeq \mathbb{R}$. Then the natural perfect pairing

$$\bigwedge^i V \times \bigwedge^{n-i} V \longrightarrow \bigwedge^n V \simeq \mathbb{R} \quad (0 \leq i \leq n)$$

gives rise to the i -th *Hodge star operator*

$$(8.4) \quad *: \bigwedge^i V \rightarrow (\bigwedge^{n-i} V)^\vee \simeq \bigwedge^{n-i} V^\vee.$$

Equivalently, for any $\omega, \eta \in \bigwedge^i V$,

$$\omega \wedge * \eta = \langle \omega, \eta \rangle_i \text{vol},$$

where $\langle \cdot, \cdot \rangle_i$ denotes the nondegenerate symmetric bilinear form on $\bigwedge^i V$ induced by

$$\langle u_1 \wedge \cdots \wedge u_i, v_1 \wedge \cdots \wedge v_i \rangle_i = \det(\langle u_k, v_\ell \rangle)$$

and $\text{vol} \in \bigwedge^n V$ is the oriented unit.

Suppose $\{e_1, \dots, e_n\}$ is a basis of V such that $\langle e_k, e_k \rangle = \pm 1$ ($1 \leq k \leq n$) and $\langle e_k, e_\ell \rangle = 0$ ($k \neq \ell$). If $e_1 \wedge \cdots \wedge e_n$ defines the orientation of V , then

$$(8.5) \quad * e_I = (-1)^{\text{neg}(I)} \varepsilon_n(I) e_{I^c} \quad \text{for } I \in \mathcal{I}_{n,i},$$

where we set $I^c := \{1, 2, \dots, n\} \setminus I$ and

$$(8.6) \quad \text{neg}(I) := |\{i \in I : \langle e_i, e_i \rangle = -1\}|,$$

$$(8.7) \quad \varepsilon(I) \equiv \varepsilon_n(I) := (-1)^{|\{(a,b) \in I \times I^c : a > b\}|} = \prod_{a \in I} \text{sgn}(I^c; a).$$

The last equality of (8.7) follows readily from the definition of $\text{sgn}(I; a)$ (see Definition 5.1). A special case of (8.5) shows $*1 = \text{vol}$. The signature $\varepsilon_n : \mathcal{I}_{n,i} \longrightarrow \{\pm 1\}$ satisfies the following formulæ.

$$(8.8) \quad \varepsilon_n(I) \varepsilon_n(I^c) = (-1)^{i(n-i)},$$

$$(8.9) \quad \varepsilon_n(I) \varepsilon_n(I \setminus \{\ell\}) = (-1)^{i+\ell} \quad \text{if } \ell \in I,$$

$$(8.10) \quad \varepsilon_n(J) \varepsilon_{n-1}(J) = 1 \quad \text{if } J \in \mathcal{I}_{n-1,i} (\subset \mathcal{I}_{n,i}).$$

From (8.5) and (8.8), we have

$$(8.11) \quad ** = (-1)^{(n-i)i} (-1)^q \text{id} \quad \text{on } \bigwedge^i V.$$

For an oriented pseudo-Riemannian manifold (X, g) of dimension n , the Hodge star operator is a linear map

$$*_X \equiv *: \mathcal{E}^i(X) \longrightarrow \mathcal{E}^{n-i}(X)$$

induced from the bijection $*_{X,x}: \bigwedge^i T_x^\vee X \longrightarrow \bigwedge^{n-i} T_x^\vee X$ for the cotangent space $T_x^\vee X$ at every $x \in X$. If $\omega, \eta \in \mathcal{E}^i(X)$ and if at least one of the supports of ω or η is compact, we set

$$(8.12) \quad (\omega, \eta) := \int_X \omega \wedge * \eta.$$

We continue a review on basic notion and results. The codifferential $d^*: \mathcal{E}^i(X) \longrightarrow \mathcal{E}^{i-1}(X)$ is given by

$$(8.13) \quad d^* = (-1)^i *^{-1} d * = (-1)^{ni+n+1} (-1)^q * d * = (-1)^{n+i+1} * d *^{-1}$$

if the signature of the pseudo-Riemannian metric is $(n-q, q)$. The second and third identities follow from (8.11). Then the codifferential d^* is the formal adjoint of the exterior derivative d in the sense that

$$(\omega, d^* \eta) = (d\omega, \eta) \quad \text{for all } \omega \in \mathcal{E}_c^i(X) \text{ and } \eta \in \mathcal{E}^{i+1}(X),$$

because $\int_X d(\omega \wedge * \eta) = 0$.

Lemma 8.1. *The following identities hold:*

$$* d d^* *^{-1} = d^* d, \quad * d^* d *^{-1} = d d^*.$$

Proof. Use (8.11) and (8.13). □

The Hodge Laplacian Δ , also known as the Laplace–de Rham operator, is a differential operator acting on differential forms is given by

$$(8.14) \quad \Delta = -(d d^* + d^* d).$$

Obviously, the Hodge star operator commutes with isometries. More generally, the Hodge star operator has a conformal covariance, which is formulated in terms of the twisted pull-back (8.2) as follows.

Lemma 8.2. *Suppose that (X, g_X) and $(X', g_{X'})$ are oriented pseudo-Riemannian manifolds of the same dimension n and that $\Phi: X \longrightarrow X'$ is a conformal map with conformal factor $\Omega \in C^\infty(X)$. Then, for any $u \in \mathbb{C}$, $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$ and $0 \leq i \leq n$, we have*

$$*_X \circ (\Phi_{u,\varepsilon}^{(i)})^* = \left(\Phi_{u-n+2i,\varepsilon+1}^{(n-i)} \right)^* \circ *_{X'} \quad \text{on } \mathcal{E}^i(X').$$

Proof. By $\Phi^* g_{X',\Phi(x)} = \Omega(x)^2 g_{X,x}$, we have the following equality:

$$(8.15) \quad *_{X,x} \circ \Phi^* = \sigma r(\Phi) \Omega(x)^{-n+2i} \Phi^* \circ *_{X',\Phi(x)} \quad \text{on } \mathcal{E}^i(X').$$

Suppose $\omega \in \mathcal{E}^i(X')$. By the definition (8.2) of $(\Phi_{u,\varepsilon}^{(i)})^*$, we have

$$*_X \circ (\Phi_{u,\varepsilon}^{(i)})^* \omega = *_X (\sigma r(\Phi)^\varepsilon \Omega^u \Phi^* \omega).$$

By (8.15), the right-hand side is equal to

$$\text{or}(\Phi)^\varepsilon \Omega^u \text{or}(\Phi) \Omega^{-u+2i} \Phi^*(*_{X'}\omega) = \left(\Phi_{u-n+2i,\varepsilon+1}^{(u-i)} \right)^* (*_{X'}\omega).$$

Hence the lemma is proved. \square

By Lemma 8.2 and (8.3), the Hodge star operator can be considered as an intertwining operator of the representations $(\varpi_{u,\varepsilon}^{(i)}, \mathcal{E}^i(X))$ of the conformal group of X :

Proposition 8.3. *Suppose that G acts conformally on an oriented pseudo-Riemannian manifold X of dimension n . Let $u \in \mathbb{C}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. Then the Hodge star operator*

$$*: \mathcal{E}^i(X) \longrightarrow \mathcal{E}^{n-i}(X)$$

intertwines the two representations $\varpi_{u,\varepsilon}^{(i)}$ and $\varpi_{u-n+2i,\varepsilon+1}^{(n-i)}$ of G , i.e.

$$* \circ \varpi_{u,\varepsilon}^{(i)}(h) = \varpi_{u-n+2i,\varepsilon+1}^{(n-i)}(h) \circ * \quad \text{for all } h \in G.$$

Example 8.4. *For $n \geq 2$ the conformal group of the standard Riemannian sphere $X = S^n$ is given by $\text{Conf}(X) \simeq O(n+1, 1)/\{\pm I_{n+2}\}$. The Hodge star operator induces an isomorphism*

$$\mathcal{E}^i(S^n)_{\lambda-i,0} \xrightarrow{\sim} \mathcal{E}^{n-i}(S^n)_{\lambda-n+i,1}$$

as $\text{Conf}(X)$ -modules by Proposition 8.3, which gives a geometric realization of the G -isomorphism between principal series representations

$$(8.16) \quad I(i, \lambda)_i \xrightarrow{\sim} I(n-i, \lambda)_i \otimes \chi_{--}$$

(see Lemma 2.2) via (2.12).

The exterior derivative d commutes with any diffeomorphism. By the conformal covariance for the Hodge star operator (Proposition 8.3), we have one for the codifferential d^* :

Lemma 8.5. *Suppose that X and X' are oriented pseudo-Riemannian manifolds of the same dimension n , and that $\Phi: X \longrightarrow X'$ is a conformal map with conformal factor $\Omega \in C^\infty(X)$.*

$$(1) \quad d_X \circ \left(\Phi_{0,\varepsilon}^{(i)} \right)^* = \left(\Phi_{0,\varepsilon}^{(i+1)} \right)^* \circ d_{X'} \quad \text{on } \mathcal{E}^i(X').$$

$$(2) \quad d_X^* \circ \left(\Phi_{n-2i,\varepsilon}^{(i)} \right)^* = \left(\Phi_{n-2i+2,\varepsilon}^{(i-1)} \right)^* \circ d_{X'}^* \quad \text{on } \mathcal{E}^i(X').$$

Proof. The first statement is obvious because the exterior derivative d commutes with any diffeomorphism. To see the second statement, we recall from (8.13) that $d^* = c * d *$ with $c := (-1)^{ni+n+1}(-1)^q$ if the signature of the pseudo-Riemannian

metric is $(n - q, q)$. By Proposition 8.3 and the first statement of this lemma, we have

$$\begin{aligned}
 d_X^* \circ \left(\Phi_{n-2i,\varepsilon}^{(i)} \right)^* &= c *_{X'} \circ d_X \circ *_{X'} \circ \left(\Phi_{n-2i,\varepsilon}^{(i)} \right)^* \\
 &= c *_{X'} \circ d_X \circ \left(\Phi_{0,\varepsilon+1}^{(n-i)} \right)^* \circ *_{X'} \\
 &= c *_{X'} \circ \left(\Phi_{0,\varepsilon+1}^{(n-i+1)} \right)^* \circ d_{X'} \circ *_{X'} \\
 &= c \left(\Phi_{n-2i+2,\varepsilon}^{(i-1)} \right)^* \circ *_{X'} \circ d_{X'} \circ *_{X'} \\
 &= \left(\Phi_{n-2i+2,\varepsilon}^{(i-1)} \right)^* \circ d_{X'}^*.
 \end{aligned}$$

Thus the lemma is proved. \square

The following proposition is immediate from Lemma 8.5.

Proposition 8.6. *Suppose X is an oriented pseudo-Riemannian manifold of dimension n , and G acts conformally on X .*

- (1) *The exterior derivative $d: \mathcal{E}^i(X) \longrightarrow \mathcal{E}^{i+1}(X)$ intertwines the two representations $\varpi_{0,\varepsilon}^{(i)}$ and $\varpi_{0,\varepsilon}^{(i+1)}$ of G for $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$.*
- (2) *The codifferential $d^*: \mathcal{E}^{i+1}(X) \longrightarrow \mathcal{E}^i(X)$ intertwines the two representations $\varpi_{n-2i-2,\varepsilon}^{(i+1)}$ and $\varpi_{n-2i,\varepsilon}^{(i)}$ of G for $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$.*

Remark 8.7. We shall prove in Section 12 that there does not exist any nonzero conformally equivariant differential operator $\mathcal{E}^i(X)_{u,\delta} \longrightarrow \mathcal{E}^{i+1}(X)_{v,\varepsilon}$ or $\mathcal{E}^{i+1}(X)_{u,\delta} \longrightarrow \mathcal{E}^i(X)_{v,\varepsilon}$ other than the differential d or the codifferential d^* (up to scalar), respectively, when X is the standard Riemannian sphere S^n .

Applying the conformal covariance of the Hodge star operator, we obtain a duality theorem for symmetry breaking operators in conformal geometry:

Theorem 8.8 (duality theorem). *Suppose (X, g) is an n -dimensional oriented pseudo-Riemannian manifold, Y is an m -dimensional submanifold such that $g|_Y$ is nondegenerate, and G' is a group acting conformally on X and leaving Y invariant. Then for any $u, v \in \mathbb{C}$, $\delta, \varepsilon \in \mathbb{Z}/2\mathbb{Z}$, and $0 \leq i \leq n$, $0 \leq j \leq m$ there is a natural bijection*

$$\text{Diff}_{G'}(\mathcal{E}^i(X)_{u,\delta}, \mathcal{E}^j(Y)_{v,\varepsilon}) \xrightarrow{\sim} \text{Diff}_{G'}(\mathcal{E}^{n-i}(X)_{u-n+2i,\delta+1}, \mathcal{E}^{m-j}(Y)_{v-m+2j,\varepsilon+1}).$$

Proof. Let $*_X$ and $*_Y$ be the Hodge star operators on (X, g) and $(Y, g|_Y)$, respectively. Then the assertion of the theorem is deduced from Proposition 8.3, summarized in

the following diagram of G' -homomorphisms:

$$\begin{array}{ccc} \mathcal{E}^i(X)_{u,\delta} & \xrightarrow[\ast_X]{\sim} & \mathcal{E}^{n-i}(X)_{u-n+2i,\delta+1} \\ \downarrow & & \downarrow \\ \mathcal{E}^j(Y)_{v,\varepsilon} & \xrightarrow[\ast_Y]{\sim} & \mathcal{E}^{m-j}(Y)_{v-m+2j,\varepsilon+1}. \end{array}$$

□

8.3. Normal derivatives under conformal transformations. Suppose that (X, g) is an oriented pseudo-Riemannian manifold of dimension n , and Y an oriented submanifold of X such that $g|_Y$ is nondegenerate. Let $G = \text{Conf}(X) \equiv \text{Conf}(X, g)$ be the group of conformal diffeomorphisms of (X, g) , and

$$G' = \text{Conf}(X; Y) := \{h \in G : hY = Y\}.$$

As in (8.1), we have group homomorphisms

$$\text{or}_X : G \longrightarrow \{\pm 1\}, \quad \text{or}_Y : G' \longrightarrow \{\pm 1\},$$

depending on whether or not the transformation preserves the orientation of X , Y , respectively.

We begin with the conformal invariance of the restriction map Rest_Y .

Lemma 8.9. *Let X and Y be oriented pseudo-Riemannian manifolds as above. Then the restriction map*

$$\text{Rest}_Y : \mathcal{E}^i(X) \longrightarrow \mathcal{E}^i(Y)$$

is a symmetry breaking operator from the representation $\varpi_{u,\delta}^{(i)}|_{G'}$ of G restricted to G' to the representation $\varpi_{u,\varepsilon}^{(i)}$ of G' for all $u \in \mathbb{C}$ if $\delta \equiv \varepsilon \equiv 0 \pmod{2}$.

Proof. We consider the condition on $(u, v; \delta, \varepsilon) \in \mathbb{C}^2 \times (\mathbb{Z}/2\mathbb{Z})^2$ such that Rest_Y intertwines $\varpi_{u,\delta}^{(i)}|_{G'}$ and $\varpi_{v,\varepsilon}^{(i)}$. For $h \in G'$ and $\eta \in \mathcal{E}^i(X)$,

$$\varpi_{v,\varepsilon}^{(i)}(h) \circ \text{Rest}_Y \eta = \text{or}_Y(h)^\varepsilon \Omega(h^{-1}, \text{Rest}_Y \cdot)^v (L_{h^{-1}})^* \text{Rest}_Y \eta,$$

$$\text{Rest}_Y \circ \varpi_{u,\delta}^{(i)}(h) \eta = \text{or}_X(h)^\delta \Omega(h^{-1}, \text{Rest}_Y \cdot)^u (L_{h^{-1}})^* \text{Rest}_Y \eta,$$

by the definition (1.1). Hence the right-hand sides of the two equalities coincide for any $h \in G'$ if $u = v$ and $\delta \equiv \varepsilon \equiv 0 \pmod{2}$. □

Suppose now that Y is of codimension one in X . Then we can define the normal vector field $N_Y(X)$ on Y such that

$$\iota_{N_Y(X)} \text{vol}_X = (-1)^{n-1} \text{vol}_Y \quad \text{on } Y,$$

where vol_X and vol_Y are the oriented volume forms of X and Y , respectively.

Example 8.10. Let $(X, Y) = (\mathbb{R}^n, \mathbb{R}^{n-1} \times \{0\})$. With the standard orientation for $Y \subset X$, the normal vector field $N_Y(X)$ is given by

$$N_Y(X) = \frac{\partial}{\partial x_n} \quad \text{on } Y,$$

because $\iota_{\frac{\partial}{\partial x_n}}(dx_1 \wedge \cdots \wedge dx_n) = (-1)^{n-1} dx_1 \wedge \cdots \wedge dx_{n-1}$.

Similarly to the pair $X \supset Y$, suppose Y' is an oriented hypersurface of a pseudo-Riemannian manifold (X', g') . Let $\Phi: X \rightarrow X'$ be a conformal map such that $\Phi(Y) \subset Y'$. We write $\Omega \equiv \Omega_\Phi \in C^\infty(X)$ for the conformal factor, namely, $\Phi^*(g_{X'}) = \Omega^2 g_X$. By a little abuse of notation, we define $\sigma_{X/Y}(\Phi) \in \{\pm 1\}$ by the identity

$$(8.17) \quad \sigma_X(\Phi) = \sigma_Y(\Phi) \sigma_{X/Y}(\Phi),$$

where we recall $\sigma_X(\Phi) \in \{\pm 1\}$ from (8.1), and $\sigma_Y(\Phi) \in \{\pm 1\}$ is defined similarly for $\Phi|_Y: Y \rightarrow Y'$. Then, we have the following:

Lemma 8.11. (1) For all $\omega \in \mathcal{E}^i(X')$, we have

$$\iota_{N_Y(X)}(\Phi^* \omega) = \sigma_{X/Y}(\Phi) \Omega \Phi^* (\iota_{N_{Y'}(X')} \omega) \quad \text{on } Y.$$

(2) For any $u \in \mathbb{C}$, we have

$$(\text{Rest}_Y \circ \iota_{N_Y(X)}) \circ \left(\Phi_{u,1}^{(i)} \right)^* = \left(\Phi_{u+1,1}^{(i-1)} \right)^* \circ (\text{Rest}_{Y'} \circ \iota_{N_{Y'}(X')}) \quad \text{on } \mathcal{E}^i(X').$$

Proof. (1) Take $p \in Y$ and local coordinates (x'_1, \dots, x'_n) on X' near $p' := \Phi(p)$ such that Y' is given locally by $x'_n = 0$ and that $\left\{ \frac{\partial}{\partial x'_1}, \dots, \frac{\partial}{\partial x'_n} \right\}$ forms an oriented orthonormal basis of $T_{p'}(X')$. We set $x_j := x'_j \circ \Phi$. Then (x_1, \dots, x_n) are local coordinates near p and the submanifold Y is given locally by $x_n = 0$. Then,

$$\left\{ \Omega(p) \frac{\partial}{\partial x_1}, \dots, \Omega(p) \frac{\partial}{\partial x_{n-2}}, \sigma_Y(\Phi) \Omega(p) \frac{\partial}{\partial x_{n-1}}, \sigma_{X/Y}(\Phi) \Omega(p) \frac{\partial}{\partial x_n} \right\}$$

is an oriented orthonormal basis of $T_p X$. We note that $\frac{\partial}{\partial x'_n}|_{p'}$ and $\sigma_{X/Y}(\Phi) \Omega(p) \frac{\partial}{\partial x_n}|_p$ are the normal vectors to Y' in X' at p' , and to Y in X at p , respectively.

Let $\omega = f dx_I$ be an i -form near p' , where $I \in \mathcal{I}_{n,i}$. Then,

$$\begin{aligned} \iota_{N_Y(X)}(\Phi^* \omega)|_p &= \sigma_{X/Y}(\Phi) f(p') \iota_{\Omega(p) \frac{\partial}{\partial x_n}|_p} dx_I \\ &= \sigma_{X/Y}(\Phi) f(p') \Omega(p) \iota_{\frac{\partial}{\partial x_n}|_p} dx_I, \\ \Omega \Phi^* (\iota_{N_{Y'}(X')} \omega)|_p &= \Omega(p) \Phi^* \left(\iota_{\frac{\partial}{\partial x'_n}|_{p'}} f(p') dx_I' \right) \\ &= f(p') \Omega(p) \Phi^* \left(\iota_{\frac{\partial}{\partial x_n}} dx_I' \right). \end{aligned}$$

Hence we have proved

$$\iota_{N_Y(X)}(\Phi^*\omega) = \sigma_{X/Y}(\Phi)\Omega\Phi^*(\iota_{N_{Y'}(X')}\omega)$$

for all $p' \in Y'$ and $\omega \in \mathcal{E}^i(X')$.

(2) We consider the condition on $(u, v; \delta, \varepsilon) \in \mathbb{C}^2 \times (\mathbb{Z}/2\mathbb{Z})^2$ such that

$$(\text{Rest}_Y \circ \iota_{N_Y(X)}) \circ (\Phi_{u,\delta}^{(i)})^* = (\Phi_{v,\varepsilon}^{(i-1)})^* \circ (\text{Rest}_{Y'} \circ \iota_{N_{Y'}(X')}) \quad \text{on } \mathcal{E}^i(X').$$

Let $\eta \in \mathcal{E}^i(X')$. Then

$$\begin{aligned} (\text{Rest}_Y \circ \iota_{N_Y(X)}) \circ (\Phi_{u,\delta}^{(i)})^* \eta &= \sigma_X(\Phi)^\delta (\text{Rest}_Y \circ \Omega)^u \text{Rest}_Y \circ \iota_{N_Y(X)}(\Phi^*\eta) \\ &= \sigma_X(\Phi)^\delta (\text{Rest}_Y \circ \Omega)^{u+1} \sigma_{X/Y}(\Phi) \text{Rest}_Y \circ \Phi^*(\iota_{N_{Y'}(X')}\eta) \end{aligned}$$

by the first statement. On the other hand,

$$(\Phi_{v,\varepsilon}^{(i-1)})^* \circ (\text{Rest}_{Y'} \circ \iota_{N_{Y'}(X')}) \eta = \sigma_{Y'}(\Phi)^\varepsilon (\text{Rest}_Y \circ \Omega)^v \text{Rest}_Y \circ \Phi^*(\iota_{N_{Y'}(X')}\eta)$$

because the conformal factor of the map $\Phi|_Y: Y \rightarrow Y'$ is given by $\text{Rest}_Y \circ \Omega$. The right-hand sides are equal if

$$u + 1 = v, \quad \sigma_X(h)^\delta \sigma_{X/Y}(h) = \sigma_{Y'}(h)^\varepsilon.$$

Hence the second statement follows from the definition (8.17) of $\sigma_{X/Y}$. \square

As an immediate consequence of Lemma 8.11 (2), we obtain:

Proposition 8.12. *Let $G = \text{Conf}(X)$ and $G' = \text{Conf}(X; Y) := \{h \in G : hY = Y\}$. Then the interior multiplication by a normal vector field*

$$\text{Rest}_Y \circ \iota_{N_Y(X)}: \mathcal{E}^i(X) \rightarrow \mathcal{E}^{i-1}(Y)$$

yields a symmetry breaking operator from the representation $\varpi_{u,\delta}^{(i)}$ of G to the representation $\varpi_{u+1,\varepsilon}^{(i-1)}$ of the subgroup G' , for all $u \in \mathbb{C}$ if $\delta \equiv \varepsilon \equiv 1 \pmod{2}$.

Remark 8.13. Alternatively, we can reduce the proof of Proposition 8.12 to Lemma 8.9 by Theorem 8.8 and by the following identity:

$$*_Y \circ \text{Rest}_Y \circ \iota_{N_Y(X)} \circ (*_X)^{-1} = \kappa \text{Rest}_Y$$

with $\kappa = \pm 1$ depending on the signature of $g(N_Y(X), N_Y(X))$. See Lemma 8.19 below for the case $(X, Y) = (\mathbb{R}^n, \mathbb{R}^{n-1})$.

8.4. Basic operators on $\mathcal{E}^i(\mathbb{R}^n)$. In this section, we assume that $Y = \mathbb{R}^{n-1}$ is the hyperplane given by $x_n = 0$ in the Euclidean space $X = \mathbb{R}^n$ equipped with the standard flat Riemannian structure, and collect some basic formulæ for operators $d, d^*, *, \iota_{N(Y)}$ and Rest_Y on differential forms $\mathcal{E}^i(\mathbb{R}^n)$ ($0 \leq i \leq n$).

By definition, the interior multiplication $\iota_{\frac{\partial}{\partial x_n}}$ is given by

$$(8.18) \quad \iota_{\frac{\partial}{\partial x_n}}(f dx_I) = \begin{cases} 0 & \text{if } n \notin I, \\ (-1)^{i-1} f dx_{I \setminus \{n\}} & \text{if } n \in I, \end{cases}$$

for $f \in C^\infty(\mathbb{R}^n)$ and $I \in \mathcal{I}_{n,i}$.

By using the notation $\text{sgn}(I; \ell)$ (see Definition 5.1), the differential d and its formal adjoint d^* (codifferential) are given by

$$(8.19) \quad d_{\mathbb{R}^n}(f dx_I) = \sum_{\ell \notin I} \text{sgn}(I; \ell) \frac{\partial f}{\partial x_\ell} dx_{I \cup \{\ell\}},$$

$$(8.20) \quad d_{\mathbb{R}^n}^*(f dx_I) = - \sum_{\ell \in I} \text{sgn}(I; \ell) \frac{\partial f}{\partial x_\ell} dx_{I \setminus \{\ell\}}.$$

Combining (8.19) and (8.20) with Lemma 5.2 (3), we have

$$(8.21) \quad d_{\mathbb{R}^n} d_{\mathbb{R}^n}^*(f dx_I) = - \sum_{p \in I} \frac{\partial^2 f}{\partial x_p^2} dx_I - \sum_{\substack{p \in I \\ q \notin I}} \text{sgn}(I; p, q) \frac{\partial^2 f}{\partial x_p \partial x_q} dx_{I \setminus \{p\} \cup \{q\}},$$

$$(8.22) \quad d_{\mathbb{R}^n}^* d_{\mathbb{R}^n}(f dx_I) = - \sum_{q \notin I} \frac{\partial^2 f}{\partial x_q^2} dx_I + \sum_{\substack{p \in I \\ q \notin I}} \text{sgn}(I; p, q) \frac{\partial^2 f}{\partial x_p \partial x_q} dx_{I \setminus \{p\} \cup \{q\}}.$$

The Laplacian $\Delta_{\mathbb{R}^n} = -(d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* + d_{\mathbb{R}^n}^* d_{\mathbb{R}^n})$ on $\mathcal{E}^i(\mathbb{R}^n)$ ((8.14)) takes the form

$$\Delta_{\mathbb{R}^n}(f dx_I) = \left(\sum_{j=1}^n \frac{\partial^2 f}{\partial x_j^2} \right) dx_I.$$

We note that the “scalar-valued” operators $\frac{\partial}{\partial x_n}$ and $\Delta_{\mathbb{R}^n} \in \text{End}(\mathcal{E}^i(\mathbb{R}^n))$ commute with any of “vector-valued” operators $*_{\mathbb{R}^n}, d_{\mathbb{R}^n}, d_{\mathbb{R}^n}^*$, and $\iota_{\frac{\partial}{\partial x_n}}$. Here are commutation relations among vector-valued operators $\mathcal{E}^i(\mathbb{R}^n) \longrightarrow \mathcal{E}^j(\mathbb{R}^n)$:

Lemma 8.14. *We have the following identities on $\mathcal{E}^i(\mathbb{R}^n)$ ($0 \leq i \leq n$).*

$$\begin{aligned}
(1) \quad & d_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}} + \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n} = \frac{\partial}{\partial x_n}. \\
(2) \quad & d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} + \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n}^* = 0. \\
(3) \quad & \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* = d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} + d_{\mathbb{R}^n}^* \frac{\partial}{\partial x_n}. \\
(4) \quad & \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n}^* d_{\mathbb{R}^n} = d_{\mathbb{R}^n}^* d_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}} - d_{\mathbb{R}^n}^* \frac{\partial}{\partial x_n}.
\end{aligned}$$

Proof. The first and second statements follow from (8.18), (8.19), and (8.20). The third and fourth statements are immediate from (1) and (2). \square

Next we deal with differential operators from i -forms on \mathbb{R}^n to j -forms on the hyperplane \mathbb{R}^{n-1} . We collect commutation relations among $d_{\mathbb{R}^n}$, $d_{\mathbb{R}^n}^*$ and $\iota_{\frac{\partial}{\partial x_n}}$ together with the restriction map $\text{Rest}_{x_n=0}$.

Lemma 8.15. *We have the following identities of operators from $\mathcal{E}^i(\mathbb{R}^n)$ to $\mathcal{E}^j(\mathbb{R}^{n-1})$.*

$$\begin{aligned}
(1) \quad & d_{\mathbb{R}^{n-1}} \circ \text{Rest}_{x_n=0} = \text{Rest}_{x_n=0} \circ d_{\mathbb{R}^n}. \\
(2) \quad & d_{\mathbb{R}^{n-1}}^* \circ \text{Rest}_{x_n=0} = \text{Rest}_{x_n=0} \circ (d_{\mathbb{R}^n}^* + \frac{\partial}{\partial x_n} \iota_{\frac{\partial}{\partial x_n}}). \\
(3) \quad & d_{\mathbb{R}^{n-1}} d_{\mathbb{R}^{n-1}}^* \circ \text{Rest}_{x_n=0} = \text{Rest}_{x_n=0} \circ \left(d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* + \frac{\partial}{\partial x_n} d_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}} \right). \\
(4) \quad & d_{\mathbb{R}^{n-1}}^* d_{\mathbb{R}^{n-1}} \circ \text{Rest}_{x_n=0} = \text{Rest}_{x_n=0} \circ \left(\frac{\partial^2}{\partial x_n^2} + d_{\mathbb{R}^n}^* d_{\mathbb{R}^n} - \frac{\partial}{\partial x_n} d_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}} \right).
\end{aligned}$$

Proof. (1) Clear. (2) Verified by (8.18) and (8.20). (3) Immediate from (1) and (2).

(4) Applying $d_{\mathbb{R}^{n-1}}^*$ to the identity (1), and using Lemma 8.14 (1), we get the fourth statement. \square

8.5. Transformation rules involving the Hodge star operator and $\text{Rest}_{x_n=0}$.

This section collects some useful formulæ involving the Hodge star operator, in particular, those for \mathbb{R}^n and its hyperplane \mathbb{R}^{n-1} , see Lemma 8.20.

We begin with basic formulæ for the conjugation by the Hodge star operator in \mathbb{R}^n .

Definition 8.16. Given an operator $T: \mathcal{E}^{n-i}(\mathbb{R}^n) \longrightarrow \mathcal{E}^{n-j}(\mathbb{R}^n)$, we define a linear operator $T^\sharp: \mathcal{E}^i(\mathbb{R}^n) \longrightarrow \mathcal{E}^j(\mathbb{R}^n)$ by

$$T^\sharp := (-1)^{n-i} *_{\mathbb{R}^n} \circ T \circ (*_{\mathbb{R}^n})^{-1}.$$

Lemma 8.17. *The correspondence $T \mapsto T^\sharp$ is given as in Table 8.1.*

TABLE 8.1. Correspondence for $T \mapsto T^\sharp$

T	$d_{\mathbb{R}^n}$	$d_{\mathbb{R}^n}^*$	$\iota_{\frac{\partial}{\partial x_n}}$	$\frac{\partial}{\partial x_n} \iota_{\frac{\partial}{\partial x_n}} + d_{\mathbb{R}^n}^*$	$-d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n}$	$d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}}$
T^\sharp	$-d_{\mathbb{R}^n}^*$	$d_{\mathbb{R}^n}$	$-dx_n \wedge$	$d_{\mathbb{R}^n} - \frac{\partial}{\partial x_n} dx_n \wedge$	$-(dx_n \wedge) \circ d_{\mathbb{R}^n} d_{\mathbb{R}^n}^*$	$-d_{\mathbb{R}^n}^* d_{\mathbb{R}^n} \circ (dx_n \wedge)$

Proof. The first two formulæ follow from the definition of the Hodge star operator, the codifferential and (8.11), and the last three follow from the first three. We thus only demonstrate the third one, namely,

$$(8.23) \quad (-1)^{n-i} *_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}} (*_{\mathbb{R}^n})^{-1} (f dx_I) = -dx_n \wedge f dx_I$$

for $f \in C^\infty(\mathbb{R}^n)$ and $I \in \mathcal{I}_{n,i}$. Obviously, both sides vanish if $n \in I$. Suppose $n \notin I$. Then,

$$(-1)^{n-i} *_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}} (*_{\mathbb{R}^n})^{-1} (f dx_I) = -\varepsilon(I^c) \varepsilon(I^c \setminus \{n\}) f dx_{I \cup \{n\}}$$

by (8.5) and (8.18), which amounts to $(-1)^{i+1} f dx_{I \cup \{n\}}$ by (8.9). Hence (8.23) is proved. \square

We introduce a linear operator $\Pi_{n-1} : \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^i(\mathbb{R}^n)$ by

$$(8.24) \quad \Pi_{n-1} := \iota_{\frac{\partial}{\partial x_n}} \circ (dx_n \wedge).$$

In the coordinates, for $f \in C^\infty(\mathbb{R}^n)$ and $I \in \mathcal{I}_{n,i}$, we have

$$(8.25) \quad \Pi_{n-1}(f dx_I) = \begin{cases} f dx_I & \text{if } n \notin I, \\ 0 & \text{if } n \in I. \end{cases}$$

Then we have

Lemma 8.18. *The following identities hold on $\mathcal{E}^i(\mathbb{R}^n)$:*

$$(8.26) \quad \text{Rest}_{x_n=0} \circ \Pi_{n-1} = \text{Rest}_{x_n=0},$$

$$(8.27) \quad (dx_n \wedge) \circ \iota_{\frac{\partial}{\partial x_n}} + \iota_{\frac{\partial}{\partial x_n}} \circ (dx_n \wedge) = \text{id},$$

$$(8.28) \quad *_{\mathbb{R}^n} \circ \Pi_{n-1} \circ (*_{\mathbb{R}^n})^{-1} = \text{id} - \Pi_{n-1}.$$

Proof. The first identity follows immediately from (8.25). A simple computation using (8.18) and (8.19) shows the second identity. To see the third identity (8.28), we apply Lemma 8.17. Then

$$\begin{aligned} -(*_{\mathbb{R}^n}) \circ \Pi_{n-1} \circ (*_{\mathbb{R}^n})^{-1} &= \left((-1)^{n-(i-1)} *_{\mathbb{R}^n} \circ \iota_{\frac{\partial}{\partial x_n}} \circ (*_{\mathbb{R}^n})^{-1} \right) \circ \left((-1)^{n-i} *_{\mathbb{R}^n} \circ (dx_n \wedge) \circ (*_{\mathbb{R}^n})^{-1} \right) \\ &= (-dx_n \wedge) \circ \iota_{\frac{\partial}{\partial x_n}}. \end{aligned}$$

Hence we get (8.28) by (8.24) and (8.27). \square

By the definition of the interior multiplication, we have the following direct sum decomposition

$$\mathcal{E}^i(\mathbb{R}^n) = \text{Image}(dx_n \wedge) \oplus \text{Ker} \left(\iota_{\frac{\partial}{\partial x_n}} \right).$$

Then the formulæ (8.25) and (8.27) show that the operators $\text{id} - \Pi_{n-1} = (dx_n \wedge) \circ \iota_{\frac{\partial}{\partial x_n}}$ and $\Pi_{n-1} = \iota_{\frac{\partial}{\partial x_n}} \circ (dx_n \wedge)$ are the first and second projections, respectively.

Next, we consider the conjugation by the two Hodge star operators $*_{\mathbb{R}^n}$ and $*_{\mathbb{R}^{n-1}}$ on \mathbb{R}^n and \mathbb{R}^{n-1} , simultaneously. For this, we observe the following basic formula.

Lemma 8.19. *We have*

$$*_{\mathbb{R}^{n-1}} \circ \text{Rest}_{x_n=0} \circ (*_{\mathbb{R}^n})^{-1} = (-1)^{k+1} \text{Rest}_{x_n=0} \circ \iota_{\frac{\partial}{\partial x_n}} \quad \text{on } \mathcal{E}^k(\mathbb{R}^n).$$

Proof. Fix $I \in \mathcal{I}_{n,k}$ and take $f(x) \equiv f(x', x_n) \in C^\infty(\mathbb{R}^n)$. We set $\omega := f(x) dx_I$. By (8.5), we have

$$(*_{\mathbb{R}^n})^{-1} \omega = \varepsilon_n(I^c) f(x) dx_{I^c},$$

and thus

$$\text{Rest}_{x_n=0} \circ (*_{\mathbb{R}^n})^{-1} \omega = \begin{cases} \varepsilon_n(I^c) f(x', 0) dx_{I^c} & \text{if } n \in I, \\ 0 & \text{otherwise.} \end{cases}$$

In turn, we obtain from (8.10)

$$*_{\mathbb{R}^{n-1}} \circ \text{Rest}_{x_n=0} \circ (*_{\mathbb{R}^n})^{-1} \omega = \begin{cases} f(x', 0) dx_{I \setminus \{n\}} & \text{if } n \in I, \\ 0 & \text{otherwise.} \end{cases}$$

Now the proposed equality follows from the identity (8.18). \square

We collect some useful formulæ involving $*_{\mathbb{R}^n}$ and $*_{\mathbb{R}^{n-1}}$. All of the operators T in the next lemma decrease the degree of forms by one.

Lemma 8.20. *Let (T, T^\flat) be a pair of linear operators $T: \mathcal{E}^{n-i}(\mathbb{R}^n) \longrightarrow \mathcal{E}^{n-i-1}(\mathbb{R}^n)$ and $T^\flat: \mathcal{E}^i(\mathbb{R}^n) \longrightarrow \mathcal{E}^i(\mathbb{R}^n)$ such that*

$$(1) \quad (T, T^\flat) = (T, -\iota_{\frac{\partial}{\partial x_n}} T^\sharp) \quad \text{with } T^\sharp \text{ the linear operator defined in Definition 8.16,}$$

or

$$(2) \quad T \text{ and } T^\flat \text{ are given in Table 8.2.}$$

Then they satisfy the following identity:

$$(8.29) \quad (-1)^{n-1} *_{\mathbb{R}^{n-1}} \circ \text{Rest}_{x_n=0} \circ T \circ (*_{\mathbb{R}^n})^{-1} = \text{Rest}_{x_n=0} \circ T^\flat.$$

TABLE 8.2. Pairs of operators (T, T^\flat) satisfying (8.29)

T	$d_{\mathbb{R}^n}^*$	$\iota_{\frac{\partial}{\partial x_n}}$	$-d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n}$	$\frac{\partial}{\partial x_n} \iota_{\frac{\partial}{\partial x_n}} + d_{\mathbb{R}^n}^*$
T^\flat	$-\iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n}$	id	$d_{\mathbb{R}^n} d_{\mathbb{R}^n}^*$	$d_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}}$

Remark 8.21. We note that T^\flat is not uniquely determined by T . For instance, $(T, T^\flat) = (\iota_{\frac{\partial}{\partial x_n}}, \Pi_{n-1})$ also satisfies (8.29), as $-\iota_{\frac{\partial}{\partial x_n}} (\iota_{\frac{\partial}{\partial x_n}})^\sharp = \Pi_{n-1}$. The choices of T^\flat in Table 8.2 are intended for simple description of differential symmetry breaking operators $\mathcal{D}_{u,\delta}^{i \rightarrow j}$, see (1.4)-(1.12).

Proof of Lemma 8.20. (1) We compose the formula

$$*\mathbb{R}^{n-1} \circ \text{Rest}_{x_n=0} \circ (*\mathbb{R}^n)^{-1} = (-1)^i \text{Rest}_{x_n=0} \circ \iota_{\frac{\partial}{\partial x_n}} \quad \text{on } \mathcal{E}^{i+1}(\mathbb{R}^n)$$

(see Lemma 8.19 (1)) with the defining relation of T^\sharp :

$$*\mathbb{R}^n \circ T \circ (*\mathbb{R}^n)^{-1} = (-1)^{n-i} T^\sharp.$$

Then we see that (8.29) is equivalent to the relation

$$(8.30) \quad \text{Rest}_{x_n=0} \circ T^\flat = -\text{Rest}_{x_n=0} \circ \iota_{\frac{\partial}{\partial x_n}} T^\sharp.$$

Hence the first statement is proved.

(2) For $T = d_{\mathbb{R}^n}^*$, we have $T^\sharp = d_{\mathbb{R}^n}$ by the second formula of Lemma 8.17, and therefore $T^\flat = -\iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n}$ satisfies (8.29).

For $T = \iota_{\frac{\partial}{\partial x_n}}$, we have $T^\sharp = -dx_n \wedge$ by the third formula of Lemma 8.17, and therefore $-\iota_{\frac{\partial}{\partial x_n}} T^\sharp = \Pi_{n-1}$ by (8.24). Hence (8.29) holds by (8.26).

For $T = -d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n}$, we have $T^\sharp = -(dx_n \wedge) \circ d_{\mathbb{R}^n} d_{\mathbb{R}^n}^*$ by the fifth formula of Lemma 8.17, and therefore $-\iota_{\frac{\partial}{\partial x_n}} T^\sharp = \Pi_{n-1} d_{\mathbb{R}^n} d_{\mathbb{R}^n}^*$. Hence (8.29) holds again by (8.26).

For $T = \frac{\partial}{\partial x_n} \iota_{\frac{\partial}{\partial x_n}} + d_{\mathbb{R}^n}^*$, we have $T^\sharp = d_{\mathbb{R}^n} - \frac{\partial}{\partial x_n} dx_n \wedge$ by the fourth formula of Lemma 8.17, and therefore

$$-\iota_{\frac{\partial}{\partial x_n}} T^\sharp = -\iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n} + \Pi_{n-1} \frac{\partial}{\partial x_n} = d_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}} + (\Pi_{n-1} - \text{id}) \frac{\partial}{\partial x_n}$$

by Lemma 8.14 (1). Hence (8.29) holds by (8.26). \square

For the operator $T = d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}}$, we also need another expression:

Lemma 8.22. *For $T = d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}}$, the following equality holds as elements in $\text{Hom}_{\mathbb{C}}(\mathcal{E}^i(\mathbb{R}^n), \mathcal{E}^i(\mathbb{R}^{n-1}))$*

$$(8.31) \quad (-1)^{n-1} *_{\mathbb{R}^{n-1}} \circ \text{Rest}_{x_n=0} \circ T \circ (*_{\mathbb{R}^n})^{-1} = d_{\mathbb{R}^{n-1}}^* d_{\mathbb{R}^{n-1}} \circ \text{Rest}_{x_n=0}.$$

Proof. By the sixth formula of Lemma 8.17, $T^\sharp = -d_{\mathbb{R}^n}^* d_{\mathbb{R}^n} \circ (dx_n \wedge)$. By (8.29) and (8.30), it suffices to show

$$(8.32) \quad \text{Rest}_{x_n=0} \circ \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n}^* d_{\mathbb{R}^n} \circ (dx_n \wedge) = d_{\mathbb{R}^{n-1}}^* d_{\mathbb{R}^{n-1}} \circ \text{Rest}_{x_n=0}.$$

By Lemma 8.14 (2), the left-hand side of (8.32) amounts to

$$-\text{Rest}_{x_n=0} \circ d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n} \circ (dx_n \wedge).$$

By Lemma 8.15 (2) and by $(\iota_{\frac{\partial}{\partial x_n}})^2 = 0$, this is equal to

$$-d_{\mathbb{R}^{n-1}}^* \text{Rest}_{x_n=0} \circ \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n} \circ (dx_n \wedge).$$

Using Lemma 8.14 (1), and by the obvious identity $\text{Rest}_{x_n=0} \circ (dx_n \wedge) = 0$, this is equal to

$$d_{\mathbb{R}^{n-1}}^* d_{\mathbb{R}^{n-1}} \text{Rest}_{x_n=0} \circ \iota_{\frac{\partial}{\partial x_n}} \circ (dx_n \wedge).$$

Now the desired equation (8.32) follows from (8.24) and (8.26). \square

8.6. Symbol maps for differential operators acting on forms. In this section, we relate matrix-valued invariant polynomials

$$\begin{aligned} H_{i \rightarrow j}^{(k)} &\in \text{Hom}_{O(N)}(\bigwedge^i(\mathbb{C}^N), \bigwedge^j(\mathbb{C}^N) \otimes \text{Pol}^k[\zeta_1, \dots, \zeta_N]), \\ \tilde{H}_{i \rightarrow j}^{(k)} &\in \text{Hom}_{O(N)}(\bigwedge^i(\mathbb{C}^N), \bigwedge^j(\mathbb{C}^N) \otimes \mathcal{H}^k(\mathbb{C}^N)) \end{aligned}$$

(see Section 5.3) with basic operators in differential geometry via the symbol map

$$\text{Symb}: \text{Diff}^{\text{const}}(\mathcal{E}^i(\mathbb{R}^N), \mathcal{E}^j(\mathbb{R}^N)) \longrightarrow \text{Hom}_{\mathbb{C}}(\bigwedge^i(\mathbb{C}^N), \bigwedge^j(\mathbb{C}^N) \otimes \text{Pol}[\zeta_1, \dots, \zeta_N]).$$

The dimension N will be taken to be $n-1$ in the next section and to be n in Chapter 12.

Lemma 8.23.

$$\begin{aligned} (1) \quad &\text{Symb}(d_{\mathbb{R}^N}) &&= H_{i \rightarrow i+1}^{(1)} \\ (2) \quad &\text{Symb}(d_{\mathbb{R}^N}^*) &&= -H_{i \rightarrow i-1}^{(1)} \\ (3) \quad &\text{Symb}(d_{\mathbb{R}^N} d_{\mathbb{R}^N}^*) &&= -H_{i \rightarrow i}^{(2)} = -\tilde{H}_{i \rightarrow i}^{(2)} - \frac{i}{N} Q_N H_{i \rightarrow i}^{(0)} \\ (4) \quad &\text{Symb}(d_{\mathbb{R}^N}^* d_{\mathbb{R}^N}) &&= -Q_N H_{i \rightarrow i}^{(0)} + H_{i \rightarrow i}^{(2)} = \tilde{H}_{i \rightarrow i}^{(2)} + \left(\frac{i}{N} - 1\right) Q_N H_{i \rightarrow i}^{(0)} \\ (5) \quad &\text{Symb}(d_{\mathbb{R}^N} d_{\mathbb{R}^N}^* + d_{\mathbb{R}^N}^* d_{\mathbb{R}^N}) &&= -Q_N H_{i \rightarrow i}^{(0)} \\ (6) \quad &\text{Symb}\left(\left(\frac{i}{N} - 1\right) d_{\mathbb{R}^N} d_{\mathbb{R}^N}^* + \frac{i}{N} d_{\mathbb{R}^N}^* d_{\mathbb{R}^N}\right) &&= \tilde{H}_{i \rightarrow i}^{(2)}. \end{aligned}$$

Proof. We compare (8.19) with (5.9), which yields the first identity. Likewise, comparing (8.20) with (5.8), we get the second identity.

The third and fourth statements follow from (8.21) and (8.22). The last two identities are now clear. \square

As a consequence of Lemma 8.23, we give a short proof of Lemma 5.5 which has been postponed.

Proof of Lemma 5.5. Since the symbol map is $O(N)$ -equivariant, and since both $d_{\mathbb{R}^n}$ and $d_{\mathbb{R}^n}^*$ commute with $O(N)$ -actions, we conclude that all the terms in the right-hand sides in Lemma 8.23 are $O(N)$ -equivariant maps.

Therefore the bilinear maps $B^{(k)}$ ($k = 0, 1, 2$) are $O(N)$ -equivariant because $\bigwedge^j(\mathbb{C}^N)$ is self-dual as an $O(N)$ -module. \square

In Proposition 5.14 we have determined the triple (i, j, k) of nonnegative integers for which the space $\text{Hom}_{O(n-1)}(\bigwedge^i(\mathbb{C}^n), \bigwedge^j(\mathbb{C}^{n-1}) \otimes \mathcal{H}^k(\mathbb{C}^{n-1}))$ is nonzero, and found an explicit basis $h_{i \rightarrow j}^{(k)}$ in (5.24) – (5.27). The next proposition describes differential operators $T_{i \rightarrow j}^{(k)}: \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^j(\mathbb{R}^n)$ such that $\text{Symb}(T_{i \rightarrow j}^{(k)}) = h_{i \rightarrow j}^{(k)}$ in all the cases.

Proposition 8.24. *We have*

Case $j = i - 2$.

$$(1) \ h_{i \rightarrow i-2}^{(1)} = \text{Symb} \left(-d_{\mathbb{R}^n}^* \circ \iota_{\frac{\partial}{\partial x_n}} \right).$$

Case $j = i - 1$.

$$(2) \ h_{i \rightarrow i-1}^{(0)} = \text{Symb} \left(\iota_{\frac{\partial}{\partial x_n}} \right).$$

$$(3) \ h_{i \rightarrow i-1}^{(1)} = \text{Symb} \left(-\Pi_{n-1} \circ d_{\mathbb{R}^n}^* - \frac{\partial}{\partial x_n} \iota_{\frac{\partial}{\partial x_n}} \right).$$

$$(4) \ h_{i \rightarrow i-1}^{(2)} = \text{Symb} \left(\left(-d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* - \frac{i-1}{n-1} \Delta_{\mathbb{R}^{n-1}} \right) \circ \iota_{\frac{\partial}{\partial x_n}} \right).$$

Case $j = i$.

$$(5) \ h_{i \rightarrow i}^{(0)} = \text{Symb}(\Pi_{n-1}).$$

$$(6) \ h_{i \rightarrow i}^{(1)} = \text{Symb} \left(d_{\mathbb{R}^n} \circ \iota_{\frac{\partial}{\partial x_n}} \right).$$

$$(7) \ h_{i \rightarrow i}^{(2)} = \text{Symb} \left(\Pi_{n-1} \circ \left(-d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* - d_{\mathbb{R}^n} \frac{\partial}{\partial x_n} \iota_{\frac{\partial}{\partial x_n}} - \frac{i}{n-1} \Delta_{\mathbb{R}^{n-1}} \right) \right).$$

Case $j = i + 1$.

$$(8) \ h_{i \rightarrow i+1}^{(1)} = \text{Symb}(\Pi_{n-1} \circ d_{\mathbb{R}^n}).$$

Proof. We shall prove the formula for $h_{i \rightarrow j}^{(k)}$ according as $k = 0, 1, 2$.

Case $k = 0$, namely, (2) and (5). We compare (8.18) with the formula for $h_{i \rightarrow i-1}^{(0)}$ in Table 5.1, and get the second identity. Likewise, comparing (8.25) with the formula for $h_{i \rightarrow i}^{(0)}$ in Table 5.1, we get the fifth identity.

Case $k = 1$, namely, (1), (3), (6), and (8).

(1) By (8.18) and (8.20), we have

$$d_{\mathbb{R}^n}^* \circ \iota_{\frac{\partial}{\partial x_n}} (f dx_I) = (-1)^i \sum_{\ell \in I \setminus \{n\}} \operatorname{sgn}(I \setminus \{n\}; \ell) \frac{\partial f}{\partial x_\ell} dx_{I \setminus \{\ell, n\}} \quad \text{for } n \in I.$$

Since $(-1)^{i-1} \operatorname{sgn}(I \setminus \{n\}; \ell) = -\operatorname{sgn}(I; \ell, n)$, we get $\operatorname{Symb} \left(d_{\mathbb{R}^n} \circ \iota_{\frac{\partial}{\partial x_n}} \right) = -h_{i \rightarrow i-2}^{(1)}$ by the formula of $h_{i \rightarrow i-2}^{(1)}$ in Table 5.1.

(3) We apply Π_{n-1} to (8.20), and get

$$(8.33) \quad \Pi_{n-1} d_{\mathbb{R}^n}^* (f dx_I) = \begin{cases} -\sum_{\ell \in I} \operatorname{sgn}(I; \ell) \frac{\partial f}{\partial x_\ell} dx_{I \setminus \{\ell\}} & (n \notin I), \\ -\operatorname{sgn}(I; n) \frac{\partial f}{\partial x_n} dx_{I \setminus \{n\}} & (n \in I). \end{cases}$$

In turn, by using (8.20), (8.18) and (8.33), we have

$$\left(-\Pi_{n-1} \circ d_{\mathbb{R}^n}^* - \frac{\partial}{\partial x_n} \iota_{\frac{\partial}{\partial x_n}} \right) (f dx_I) = \begin{cases} \sum_{\ell \in I} \operatorname{sgn}(I; \ell) \frac{\partial f}{\partial x_\ell} dx_{I \setminus \{\ell\}} & (n \notin I), \\ 0 & (n \in I). \end{cases}$$

Comparing this with the formula for $h_{i \rightarrow i-1}^{(1)}$ in Table 5.1 again, we get the third identity. The proofs for (6) and (8) are similar, and we omit them.

Case $k = 2$, namely, (4) and (7). Let us prove (4). It follows from Lemma 8.23 (3) and Proposition 8.24 (2) that

$$\operatorname{Symb} \left(-d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \circ \iota_{\frac{\partial}{\partial x_n}} - \frac{i-1}{n-1} \Delta_{\mathbb{R}^{n-1}} \iota_{\frac{\partial}{\partial x_n}} \right) = H_{i-1 \rightarrow i-1}^{(2)} \circ h_{i \rightarrow i-1}^{(0)} - \frac{i-1}{n-1} Q_{n-1}(\zeta') h_{i \rightarrow i-1}^{(0)}.$$

By the definitions (5.11) and (5.25), this amounts to

$$\tilde{H}_{i-1 \rightarrow i-1}^{(2)} \circ \operatorname{pr}_{i \rightarrow i-1} = h_{i \rightarrow i-1}^{(2)}.$$

The case (7) is similar. □

9. IDENTITIES OF SCALAR-VALUED DIFFERENTIAL OPERATORS \mathcal{D}_ℓ^μ

In this chapter, we derive identities for the (scalar-valued) differential operators \mathcal{D}_ℓ^μ (see (1.2) for the definition) systematically from those for the Gegenbauer polynomials given in Appendix. We note that some of the formulæ here were previously known up to the restriction map $\text{Rest}_{x_n=0}$, see [11, 15, 19, 22].

Using these identities together with the results of Chapter 8, we study matrix-valued symmetry breaking operators $\mathcal{D}_{u,a}^{i \rightarrow j}$ in details. In particular, the condition for the vanishing of the operators $\mathcal{D}_{u,a}^{i \rightarrow i-1}$ and $\mathcal{D}_{u,a}^{i \rightarrow i}$ (Proposition 1.4) is proved in Section 9.3, and the identity (1.4) = (1.5) about the two expressions of $\mathcal{D}_{u,a}^{i \rightarrow i-1}$ is proved in Section 9.4. Various functional identities among $\mathcal{D}_{u,a}^{i \rightarrow j}$ are proved in Chapter 13.

9.1. Homogeneous polynomial inflation I_a . Suppose $a \in \mathbb{N}$. For $g(t) \in \text{Pol}_a[t]_{\text{even}}$ (see (4.5)), we define a polynomial of two variables x and y (a -inflated polynomial of g) by

$$(9.1) \quad I_a g(x, y) = x^{\frac{a}{2}} g\left(\frac{y}{\sqrt{x}}\right).$$

Notice that $(I_a g)(x^2, y)$ is a homogeneous polynomial of x and y of degree a .

By definition, we have

$$(9.2) \quad I_{a+1}(tg(t))(x, y) = y(I_a g)(x, y),$$

$$(9.3) \quad (I_{a+2}g)(x, y) = x(I_a g)(x, y).$$

We recall $Q_{n-1}(\zeta') = \zeta_1^2 + \cdots + \zeta_{n-1}^2$ for $\zeta' = (\zeta_1, \dots, \zeta_{n-1})$, and from (4.4) that $(T_a g)(\zeta) = Q_{n-1}(\zeta')^{\frac{a}{2}} g\left(\frac{\zeta_n}{\sqrt{Q_{n-1}(\zeta')}}\right)$ is a homogeneous polynomial of n -variables $\zeta = (\zeta_1, \dots, \zeta_{n-1}, \zeta_n)$ of degree a . By definition, we have the following identity:

$$(9.4) \quad (T_a g)(\zeta) = I_a g(Q_{n-1}(\zeta'), \zeta_n).$$

If we substitute the differential operators $\Delta_{\mathbb{R}^{n-1}}$ and $\frac{\partial}{\partial x_n}$ into $I_a g(x, y)$, we get a homogeneous differential operator $I_a g\left(\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}\right)$ of order a . It then follows from (9.4) that its symbol (see (3.3)) is given by

$$(9.5) \quad \text{Symb}\left(I_a g\left(\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}\right)\right) = T_a g.$$

We recall from (1.2) that $\mathcal{D}_a^\mu = (I_a \tilde{C}_a^\mu)\left(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}\right)$ is a homogeneous differential operator on \mathbb{R}^n of order a , where $\tilde{C}_a^\mu(t)$ is the renormalized Gegenbauer polynomial (see (14.3)). Then its symbol is given as follows:

Lemma 9.1. $\text{Symb}(\mathcal{D}_a^\mu) = e^{-\frac{\pi\sqrt{-1}a}{2}} T_a \left(\tilde{C}_a^\mu \left(e^{\frac{\pi\sqrt{-1}}{2}} \cdot \right) \right).$

Proof. Suppose $g(t) \in \text{Pol}_a[t]_{\text{even}}$ is of the form $g(t) = e^{-\frac{\pi\sqrt{-1}a}{2}} \varphi(e^{\frac{\pi\sqrt{-1}}{2}} t)$ with $\varphi(s) \in \text{Pol}_a[s]_{\text{even}}$. By definition we have

$$I_a g(x, y) = I_a \varphi(-x, y),$$

and thus $I_a g(\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}) = I_a \varphi(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n})$. In turn, $\text{Symb} \left(I_a \varphi(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}) \right) = T_a g$ by (9.5). Hence Lemma follows. \square

9.2. Identities among Juhl's conformally covariant differential operators.

The composition $\text{Rest}_{x_n=0} \circ \mathcal{D}_a^\mu: C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^{n-1})$ is a conformally covariant differential operator, which we refer to as Juhl's operator. In this section we collect identities for the scalar-valued differential operators \mathcal{D}_a^μ that hold before taking the restriction operator $\text{Rest}_{x_n=0}$:

- three-term relations for general parameter μ (Proposition 9.2)
- factorization identities for integral parameter μ (Proposition 9.3, Lemma 9.4).

Proposition 9.2. *Let $a \in \mathbb{N}$, $\mu \in \mathbb{C}$, and $\gamma(\mu, a)$ be defined as in (1.3). Then we have*

$$(9.6) \quad \mathcal{D}_{a-2}^{\mu+1} \Delta_{\mathbb{R}^{n-1}} + \gamma(\mu, a) \mathcal{D}_{a-1}^{\mu+1} \frac{\partial}{\partial x_n} = \frac{a}{2} \mathcal{D}_a^\mu.$$

$$(9.7) \quad \mathcal{D}_{a-2}^{\mu+1} \frac{\partial}{\partial x_n} - \gamma(\mu, a) \mathcal{D}_{a-1}^{\mu+1} + \gamma(\mu - \frac{1}{2}, a) \mathcal{D}_{a-1}^\mu = 0.$$

$$(9.8) \quad \mathcal{D}_{a-2}^{\mu+1} \Delta_{\mathbb{R}^n} + \left(\mu - \frac{1}{2} \right) \mathcal{D}_a^\mu = \left(\mu + \left[\frac{a}{2} \right] - \frac{1}{2} \right) \mathcal{D}_a^{\mu-1}.$$

$$(9.9) \quad \gamma(\mu, a) \mathcal{D}_{a-1}^{\mu+1} \Delta_{\mathbb{R}^n} + \left(\mu - \frac{1}{2} \right) \mathcal{D}_a^\mu \frac{\partial}{\partial x_n} = \frac{1}{2} (a+1) \gamma(\mu - \frac{1}{2}, a) \mathcal{D}_{a+1}^{\mu-1}.$$

$$(9.10) \quad (\mu + a) \mathcal{D}_a^\mu - \mathcal{D}_{a-2}^{\mu+1} \Delta_{\mathbb{R}^{n-1}} = \left(\mu + \left[\frac{a+1}{2} \right] \right) \mathcal{D}_a^{\mu+1}.$$

Proof. By using (9.2) and (9.3), we see that these three-term relations for $\mathcal{D}_a^\mu = \left(I_a \tilde{C}_a^\mu \right) \left(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n} \right)$ are derived from those for Gegenbauer polynomials $\tilde{C}_a^\mu(z)$ that will be proved in Chapter 14 (Appendix). The correspondence is given in the following table:

The (scalar-valued) differential operator \mathcal{D}_ℓ^μ for specific parameter μ and ℓ may be written as the product of another operator $\mathcal{D}_{\ell'}^{\mu'}$ and the Laplacian $\Delta_{\mathbb{R}^n}$ (or $\Delta_{\mathbb{R}^{n-1}}$).

Identities for \mathcal{D}_a^μ	(9.6)	(9.7)	(9.8)	(9.9)	(9.10)
Identities for \tilde{C}_a^μ	(14.19)	(14.21)	(14.17)	(14.16)	(14.15)

□

For example,

$$\mathcal{D}_3^\mu = \Delta_{\mathbb{R}^{n-1}} \mathcal{D}_1^\mu \text{ if } \mu = -2; \quad \mathcal{D}_3^\mu = \Delta_{\mathbb{R}^n} \mathcal{D}_1^{1-\mu} \text{ if } \mu = -\frac{1}{2}.$$

We collect such factorization identities as follows.

For $a \in \mathbb{N}$ and $\ell \in \mathbb{N}_+$, we recall from (1.13) that $K_{\ell,a} := \prod_{k=1}^{\ell} \left(\left\lfloor \frac{a}{2} \right\rfloor + k \right)$ is a positive integer. For $\ell = 0$, we set $K_{\ell,a} = 1$.

Proposition 9.3. *Let $a, \ell \in \mathbb{N}$. Then*

$$\begin{aligned} (1) \quad & K_{\ell,a} \mathcal{D}_{a+2\ell}^{\frac{1}{2}-\ell} = \mathcal{D}_a^{\frac{1}{2}+\ell} \Delta_{\mathbb{R}^n}^\ell. \\ (2) \quad & K_{\ell,a} \mathcal{D}_{a+2\ell}^{-a-\ell} = \mathcal{D}_a^{-a-\ell} \Delta_{\mathbb{R}^{n-1}}^\ell. \end{aligned}$$

Proof. According to definition (9.1) for every $\ell \in \mathbb{N}$ we have

$$(9.11) \quad I_{a+2\ell} \left((z^2 - 1)^\ell g \right) (x, y) = (y^2 - x)^\ell (I_a g)(x, y).$$

Thus, applying $I_{a+2\ell}$ to the identity (14.23) in Proposition 14.11 we get (1).

Similarly, applying $I_{a+2\ell}$ to (14.22) and using (9.3) we get (2) and conclude the proof. □

Analogous formulæ are derived from Proposition 9.3, and will be used in the proof of Theorems 13.1 and 13.2.

Lemma 9.4. *Let $a \in \mathbb{N}$ and $\ell \in \mathbb{N}_+$.*

$$\begin{aligned} (1) \quad & K_{\ell,a} \mathcal{D}_{a+2\ell-2}^{-\ell+\frac{3}{2}} = \left(\ell + \left\lfloor \frac{a}{2} \right\rfloor \right) \mathcal{D}_a^{\ell-\frac{1}{2}} \Delta_{\mathbb{R}^n}^{\ell-1}. \\ (2) \quad & 4K_{\ell,a} \gamma \left(\ell + \frac{1}{2}, a-1 \right) \gamma \left(-\ell + \frac{1}{2}, a+2\ell \right) \mathcal{D}_{a+2\ell-1}^{\frac{3}{2}-\ell} = (a+1)(a+2\ell) \mathcal{D}_{a+1}^{\ell-\frac{1}{2}} \Delta_{\mathbb{R}^n}^\ell. \\ (3) \quad & K_{\ell,a} \mathcal{D}_{a+2\ell-2}^{-a-\ell+1} = \left(\ell + \left\lfloor \frac{a}{2} \right\rfloor \right) \mathcal{D}_a^{-a-\ell+1} \Delta_{\mathbb{R}^{n-1}}^{\ell-1}. \\ (4) \quad & \gamma(-a, a) K_{\ell,a} \mathcal{D}_{a+2\ell-1}^{-a-\ell+1} = \gamma(-a-\ell, a) \mathcal{D}_{a-1}^{-a-\ell+1} \Delta_{\mathbb{R}^{n-1}}^\ell. \end{aligned}$$

Proof. (1) Apply Proposition 9.3 (1) with ℓ replaced by $\ell-1$.

(2) We again apply Proposition 9.3 (2) with a replaced by $a+1$ and ℓ replaced by $\ell-1$ this time. Then the assertion follows from the identity below

$$(9.12) \quad \frac{K_{\ell,a}}{K_{\ell-1,a+1}} = \frac{(a+1)(a+2\ell)}{4\gamma \left(\ell + \frac{1}{2}, a-1 \right) \gamma \left(-\ell + \frac{1}{2}, a+2\ell \right)}.$$

The proof of (9.12) is elementary, and we omit it.

Identities (3) and (4) follow from Proposition 9.3 (2) by similar argument as we used for cases (1) and (2) above. We also use an elementary formula

$$\frac{K_{\ell,a-1}}{K_{\ell,a}} = \frac{\gamma(-a, a)}{\gamma(-a - \ell, a)}.$$

□

In the rest of this chapter, we apply the three-term relations given in Proposition 9.2.

9.3. Proof of Proposition 1.4. Given a linear operator $T : \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^j(\mathbb{R}^{n-1})$, we define the “matrix component” T_{IJ} for $I \in \mathcal{I}_{n,i}$ and $J \in \mathcal{I}_{n-1,j}$ by the identity:

$$T(fdx_I) = \sum_{J \in \mathcal{I}_{n-1,j}} (T_{IJ}f)dx_J.$$

If T is a differential operator, so is $T_{IJ} : C^\infty(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^{n-1})$.

We find the (I, J) -component of the symmetry breaking operator $\mathcal{D}_{u,a}^{i \rightarrow i-1} : \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^{i-1}(\mathbb{R}^{n-1})$ introduced in (1.4) as follows:

Lemma 9.5. *For $I \in \mathcal{I}_{n,i}$ and $J \in \mathcal{I}_{n-1,i-1}$, we consider*

- Case 1. $n \in I, J = I \setminus \{n\}$,
- Case 2. $n \in I, |J \setminus I| = 1$, say $I = K \cup \{p, n\}, J = K \cup \{q\}$,
- Case 3. $n \notin I, J \subset I$, say $I = J \cup \{p\}$.

Let $\mu := u + i - \frac{1}{2}(n-1)$. Then the matrix component $(D_{u,a}^{i \rightarrow i-1})_{IJ}$ is given as

Case 1. $-\mathcal{D}_{a-2}^{\mu+1} \sum_{p \in I^c} \frac{\partial^2}{\partial x_p^2} + \frac{1}{2}(a + u + 2i - n)\mathcal{D}_a^\mu$,

Case 2. $(-1)^{i-1} \text{sgn}(I; p, q)\mathcal{D}_{a-2}^{\mu+1}$,

Case 3. $\text{sgn}(I; p)\gamma(\mu, a)\mathcal{D}_{a-1}^{\mu+1}$,

followed by the restriction map $\text{Rest}_{x_n=0}$. Here $I^c = \{1, 2, \dots, n\} \setminus I$ in Case 1. Otherwise, the (I, J) component $(D_{u,a}^{i \rightarrow i-1})_{IJ}$ vanishes.

Proof. We recall from (1.4) that

$$\mathcal{D}_{u,a}^{i \rightarrow i-1} = \text{Rest}_{x_n=0} \circ \left(-\mathcal{D}_{a-2}^{\mu+1} d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} - \gamma(\mu, a)\mathcal{D}_{a-1}^{\mu+1} d_{\mathbb{R}^n}^* + \frac{1}{2}(u + 2i - n)\mathcal{D}_a^\mu \iota_{\frac{\partial}{\partial x_n}} \right)$$

We begin by computing the (I, J) -components of the basis elements $\text{Rest}_{x_n=0} \circ d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}}$, $\text{Rest}_{x_n=0} \circ d_{\mathbb{R}^n}^*$, and $\text{Rest}_{x_n=0} \iota_{\frac{\partial}{\partial x_n}}$.

It follows from (8.18), (8.20), and (8.21) that (I, J) -components of these operators are given as the entries in the table below, followed by the restriction map $\text{Rest}_{x_n=0}$:

	$(\text{Rest}_{x_n=0} \circ d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}})_{IJ}$	$(\text{Rest}_{x_n=0} d_{\mathbb{R}^n}^*)_{IJ}$	$(\text{Rest}_{x_n=0} \circ \iota_{\frac{\partial}{\partial x_n}})_{IJ}$
Case 1	$(-1)^i \sum_{p \in I \setminus \{n\}} \frac{\partial^2}{\partial x_p^2}$	$(-1)^i \frac{\partial}{\partial x_n}$	$(-1)^{i-1}$
Case 2	$(-1)^i \text{sgn}(I; p, q) \frac{\partial^2}{\partial x_p \partial x_q}$	0	0
Case 3	0	$-\text{sgn}(I; p) \frac{\partial}{\partial x_p}$	0.

Then Cases 2 and 3 of the lemma follow from (1.4). In Case 1, the (I, J) -component of $\mathcal{D}_{u,a}^{i \rightarrow i-1}$ is given by

$$(-1)^{i-1} \text{Rest}_{x_n=0} \circ \left(\mathcal{D}_{a-2}^{\mu+1} \sum_{p \in I \setminus \{n\}} \frac{\partial^2}{\partial x_p^2} + \gamma(\mu, a) \mathcal{D}_{a-1}^{\mu+1} \frac{\partial}{\partial x_n} + \frac{1}{2}(u + 2i - n) \mathcal{D}_a^\mu \right),$$

which amounts to

$$(-1)^{i-1} \text{Rest}_{x_n=0} \circ \left(\frac{1}{2}(a + u + 2i - n) \mathcal{D}_a^\mu + \mathcal{D}_{a-2}^{\mu+1} \left(\sum_{p \in I \setminus \{n\}} \frac{\partial^2}{\partial x_p^2} - \Delta_{\mathbb{R}^{n-1}} \right) \right)$$

by the three-term relation (9.6) for \mathcal{D}_a^μ . Thus the lemma is proved. \square

Lemma 9.5 will be used for the proof of Proposition 1.4 (1). We may deduce Proposition 1.4 (2) from Proposition 1.4 (1) by the duality (10.6), however, we give explicit formulæ for the matrix components of $\mathcal{D}_{u,a}^{i \rightarrow i}$ for later purpose.

Lemma 9.6. *For $I \in \mathcal{I}_{n,i}$ and $J \in \mathcal{I}_{n-1,i}$, we consider*

Case 1. $n \notin I, J = I$.

Case 2. $n \notin I, |J \setminus I| = 1$, say $I = K \cup \{p\}, J = K \cup \{q\}$.

Case 3. $n \in I, |J \setminus I| = 1$, say $I = K \cup \{n\}, J = K \cup \{q\}$.

Let $\mu := u + i - \frac{n-1}{2}$. Then the matrix component $(\mathcal{D}_{u,a}^{i \rightarrow i})_{IJ}$ is given as

Case 1. $-\mathcal{D}_{a-2}^{\mu+1} \sum_{p \in I} \frac{\partial^2}{\partial x_p^2} + \frac{1}{2}(u + a) \mathcal{D}_a^\mu,$

Case 2. $-\text{sgn}(I; p, q) \mathcal{D}_{a-2}^{\mu+1} \frac{\partial^2}{\partial x_p \partial x_q},$

Case 3. $-\text{sgn}(I; q, n) \gamma(\mu, a) \mathcal{D}_{a-1}^{\mu+1} \frac{\partial}{\partial x_q},$

followed by the restriction map $\text{Rest}_{x_n=0}$. Otherwise, the (I, J) -component $(\mathcal{D}_{u,a}^{i \rightarrow i})_{IJ}$ is equal to zero.

Proof. From the expressions (8.21), (8.18) and (8.19), we have:

	$(\text{Rest}_{x_n=0} \circ d_{\mathbb{R}^n}^* d_{\mathbb{R}^n}^*)_{IJ}$	$(\text{Rest}_{x_n=0} \circ d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}})_{IJ}$
Case 1	$-\text{Rest}_{x_n=0} \circ \sum_{p \in I} \frac{\partial^2}{\partial x_p^2}$	0
Case 2	$-\text{sgn}(I; p, q) \text{Rest}_{x_n=0} \circ \frac{\partial^2}{\partial x_p \partial x_q}$	0
Case 3	$-\text{sgn}(I; q, n) \text{Rest}_{x_n=0} \circ \frac{\partial^2}{\partial x_q \partial x_n}$	$(-1)^{i-1} \text{sgn}(I; q) \text{Rest}_{x_n=0} \circ \frac{\partial}{\partial x_q}$

Then Cases 1 and 2 of the lemma follow from (1.6). In Case 3, we also use the identity $\text{sgn}(I; q, n) = (-1)^{i-1} \text{sgn}(I; q)$ and the three-term relation (9.7). \square

We are ready to complete the proof of Proposition 1.4.

Proof of Proposition 1.4. (1). Suppose $i = n$. Then, only Case 1 in Lemma 9.5 occurs. In this case $I^c = \emptyset$. Thus $\mathcal{D}_{u,a}^{n \rightarrow n-1} = 0$ if and only if $a + u + 2i - n = 0$, equivalently, $u = -n - a$.

Suppose $1 \leq i \leq n-1$. Then Cases 1 and 3 in Lemma 9.5 occur, and Case 2 occurs if $2 \leq i \leq n-1$.

First, we see from Lemma 9.5 that $(\mathcal{D}_{u,a}^{i \rightarrow i-1})_{IJ} = 0$ in Case 1 if and only if $\mathcal{D}_{a-2}^{\mu+1} = 0$ and $a + u + 2i - n = 0$, equivalently, $n - u - 2i = a \in \{0, 1\}$.

Second, $(\mathcal{D}_{u,a}^{i \rightarrow i-1})_{IJ} = 0$ in Case 3 if and only if $\gamma(\mu, a) \mathcal{D}_{a-1}^{\mu+1} = 0$. This happens if and only if $a = 0$ because $a \in \{0, 1\}$. Hence $\mathcal{D}_{u,a}^{i \rightarrow i-1} = 0$ implies that $(u, a) = (n - 2i, 0)$. The converse statement also holds because $(\mathcal{D}_{u,a}^{i \rightarrow i-1})_{IJ}$ vanishes in Case 2 if $a = 0$. Thus Proposition 1.4 (1) is proved.

(2). The proof of Proposition 1.4 (2) is similar to the one of (1) by using Lemma 9.6, and we omit it. \square

9.4. Two expressions of $\mathcal{D}_{u,a}^{i \rightarrow i-1}$. In this section, we prove in Proposition 9.9 the identity (1.4) = (1.5) for the two expressions of the differential operator $\mathcal{D}_{u,a}^{i \rightarrow i-1}: \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^{i-1}(\mathbb{R}^{n-1})$ by using the three-term relations that we established in Section 9.2.

In order to prove the identity (1.4) = (1.5), we begin with the relationship between the following two triples of matrix-valued differential operators

$$\{d_{\mathbb{R}^n}^* d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}}, d_{\mathbb{R}^n}^*, \iota_{\frac{\partial}{\partial x_n}}\} \quad \text{and} \quad \{-d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n}, \frac{\partial}{\partial x_n} \iota_{\frac{\partial}{\partial x_n}} + d_{\mathbb{R}^n}^*, \iota_{\frac{\partial}{\partial x_n}}\}$$

that map $\mathcal{E}^i(\mathbb{R}^n)$ to $\mathcal{E}^{i-1}(\mathbb{R}^n)$.

Lemma 9.7. Suppose A, B, C, P, Q and R are scalar-valued differential operators on \mathbb{R}^n satisfying

$$(9.13) \quad P = -A, \quad Q = B - A \frac{\partial}{\partial x_n}, \quad R = -A \Delta_{\mathbb{R}^{n-1}} - B \frac{\partial}{\partial x_n} + C.$$

Then

$$(9.14) \quad Ad_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} + Bd_{\mathbb{R}^n}^* + C\iota_{\frac{\partial}{\partial x_n}} = P(-d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n}) + Q(\frac{\partial}{\partial x_n} \iota_{\frac{\partial}{\partial x_n}} + d_{\mathbb{R}^n}^*) + R\iota_{\frac{\partial}{\partial x_n}}.$$

Proof. It follows from Lemma 8.14 (1) and (8.14) that

$$-d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n} = d_{\mathbb{R}^n}^* d_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}} - \frac{\partial}{\partial x_n} d_{\mathbb{R}^n}^* = -d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} - \frac{\partial}{\partial x_n} d_{\mathbb{R}^n}^* - \Delta_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}}.$$

Hence the right-hand side of (9.14) is equal to

$$-Pd_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} + (-P\frac{\partial}{\partial x_n} + Q)d_{\mathbb{R}^n}^* + (-P\Delta_{\mathbb{R}^n} + Q\frac{\partial}{\partial x_n} + R)\iota_{\frac{\partial}{\partial x_n}}.$$

Thus the equality (9.14) holds if

$$A = -P, \quad B = -P\frac{\partial}{\partial x_n} + Q, \quad C = -P\Delta_{\mathbb{R}^n} + Q\frac{\partial}{\partial x_n} + R,$$

or equivalently if (9.13) is satisfied. \square

Lemma 9.8. *Suppose $\mu \in \mathbb{C}$ and $a \in \mathbb{N}$. Then we have the following identity as linear operators from $\mathcal{E}^i(\mathbb{R}^n)$ to $\mathcal{E}^{i-1}(\mathbb{R}^n)$:*

$$\begin{aligned} & -\mathcal{D}_{a-2}^{\mu+1} d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} - \gamma(\mu, a) \mathcal{D}_{a-1}^{\mu+1} d_{\mathbb{R}^n}^* + \frac{1}{2}(\mu + i - \frac{n+1}{2}) \mathcal{D}_a^\mu \iota_{\frac{\partial}{\partial x_n}} \\ = & -\mathcal{D}_{a-2}^{\mu+1} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n} - \gamma(\mu - \frac{1}{2}, a) \mathcal{D}_{a-1}^\mu (\frac{\partial}{\partial x_n} \iota_{\frac{\partial}{\partial x_n}} + d_{\mathbb{R}^n}^*) + \frac{1}{2}(\mu + i - \frac{n+1}{2} + a) \mathcal{D}_a^\mu \iota_{\frac{\partial}{\partial x_n}}. \end{aligned}$$

Proof. By Lemma 9.7 with

$$A = -\mathcal{D}_{a-2}^{\mu+1}, \quad B = -\gamma(\mu, a) \mathcal{D}_{a-1}^{\mu+1}, \quad C = \frac{1}{2}(\mu + i - \frac{n+1}{2}) \mathcal{D}_a^\mu,$$

the proof of Lemma 9.8 reduces to the following identities

$$\begin{aligned} & \mathcal{D}_{a-2}^{\mu+1} \frac{\partial}{\partial x_n} - \gamma(\mu, a) \mathcal{D}_{a-1}^{\mu+1} = -\gamma(\mu - \frac{1}{2}, a) \mathcal{D}_{a-1}^\mu, \\ & \mathcal{D}_{a-2}^{\mu+1} \Delta_{\mathbb{R}^{n-1}} + \gamma(\mu, a) \mathcal{D}_{a-1}^{\mu+1} \frac{\partial}{\partial x_n} + \frac{1}{2}(\mu + i - \frac{n+1}{2}) \mathcal{D}_a^\mu = \frac{1}{2}(\mu + i - \frac{n+1}{2} + a) \mathcal{D}_a^\mu. \end{aligned}$$

These are nothing but the three-term relations among the operators \mathcal{D}_ℓ^λ that we proved in (9.7) and (9.6), respectively. \square

We are ready to prove the second expression (1.5) of $\mathcal{D}_{u,a}^{i \rightarrow i-1}$.

Proposition 9.9. *As operators $\mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^{i-1}(\mathbb{R}^{n-1})$, we have (1.4) = (1.5).*

Proof. It follows from Lemma 8.15 (2) that

$$\text{Rest}_{x_n=0} \circ \mathcal{D}_{a-1}^\mu \left(\frac{\partial}{\partial x_n} \iota_{\frac{\partial}{\partial x_n}} + d_{\mathbb{R}^n}^* \right) = d_{\mathbb{R}^{n-1}}^* \circ \text{Rest}_{x_n=0} \circ \mathcal{D}_{a-1}^\mu.$$

Hence the proposition follows from Lemma 9.8 composed by $\text{Rest}_{x_n=0}$. \square

By the expression (1.4), the symmetry breaking operator $\mathcal{D}_{u,a}^{i \rightarrow i-1}$ takes a simpler form when $i = 1$:

$$\mathcal{D}_{u,a}^{1 \rightarrow 0} = \text{Rest}_{x_n=0} \left(-\gamma \left(u - \frac{n-3}{2}, a \right) \mathcal{D}_{a-1}^{u-\frac{n-5}{2}} d_{\mathbb{R}^n}^* + \frac{1}{2} (u+2-n) \mathcal{D}_a^{u-\frac{n-3}{2}} \iota_{\frac{\partial}{\partial x_n}} \right)$$

because

$$d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} = 0 \quad \text{on } \mathcal{E}^1(\mathbb{R}^n),$$

and so the first term of (1.4) vanishes. On the other hand, by the expression (1.5), we see that the symmetry breaking operator $\mathcal{D}_{u,a}^{i \rightarrow i-1}$ takes a simpler form when $i = n$:

$$(9.15) \quad \mathcal{D}_{u,a}^{n \rightarrow n-1} = \frac{1}{2} (u+n+a) \text{Rest}_{x_n=0} \circ \mathcal{D}_a^{u+\frac{n+1}{2}} \iota_{\frac{\partial}{\partial x_n}},$$

since both the operators

$$-d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n} \quad \text{and} \quad \text{Rest}_{x_n=0} \circ \left(\frac{\partial}{\partial x_n} \iota_{\frac{\partial}{\partial x_n}} + d_{\mathbb{R}^n}^* \right) \quad (= d_{\mathbb{R}^{n-1}}^* \circ \text{Rest}_{x_n=0})$$

in the first and third terms of (1.5) vanish on $\mathcal{E}^n(\mathbb{R}^n)$. This operator is dual (via the Hodge star operator) to the symmetry breaking operator $\mathcal{D}_{u+2i-n,a}^{0 \rightarrow 0}$ (Juhl's operator) for functions (see Section 10.4).

10. CONSTRUCTION OF DIFFERENTIAL SYMMETRY BREAKING OPERATORS

We proved in Proposition 5.19 that there exist nonzero differential symmetry breaking operators from the G -representation $I(i, \lambda)_\alpha$ to the G' -representation $J(j, \nu)_\beta$ only if

$$j \in \{i-2, i-1, i, i+1\}.$$

In this chapter, we complete the proof of Theorem 2.9 which provides explicit formulæ of these symmetry breaking operators. The formulæ are given in the flat picture (2.7), namely, as differential operators $\mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^j(\mathbb{R}^{n-1})$.

By the F-method (see Fact 3.3), we have a natural bijection (see (5.31))

$$(10.1) \quad \text{Diff}_{G'}(I(i, \lambda)_\alpha, J(j, \nu)_\beta) \simeq \text{Sol}(\mathfrak{n}_+; \sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(j)}),$$

where the right-hand side consists of (vector-valued) polynomial solutions to the F-sytem. In the previous chapters, we determined explicitly these polynomial when $j = i-1$ and $i+1$ (see Theorems 6.1 and 7.3, respectively). Then the proof for Theorem 2.9 is divided into the following two parts:

- For $j = i-1$ and $i+1$, we translate these polynomial solutions into geometric operators acting on differential forms via the symbol map according to the F-method. We show that the resulting symmetry breaking operators coincide with $\tilde{\mathbb{C}}_{\lambda, \nu}^{i, i-1}$ and $\tilde{\mathbb{C}}_{\lambda, \nu}^{i, i+1}$, respectively.
- For $j = i-2$ and i , we use the duality theorem of symmetry breaking operators (Theorem 2.7).

This completes the proof of Theorem 2.9. In the next chapter, we shall derive Theorems 1.5-1.8 from Theorem 2.9.

10.1. Proof of Theorem 2.9 in the case $j = i-1$. In this section, we give a proof of Theorem 2.9 in the case $j = i-1$. Suppose that we are in Case 2 of Theorem 2.8, namely,

$$1 \leq i \leq n, a := \nu - \lambda \in \mathbb{N} \text{ and } \beta - \alpha \equiv a \pmod{2}.$$

Let (g_0, g_1, g_2) be the triple of the nonzero polynomials given in Theorem 6.1 so that

$$\text{Sol}(\mathfrak{n}_+; \sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(i-1)}) = \mathbb{C} \sum_{k=0}^2 (T_{a-k} g_k) h_{i \rightarrow i-1}^{(k)}.$$

We recall that $g_1 = g_2 = 0$ if $i = n$ or $\lambda = \nu$. By the isomorphism (10.1), the generator $\sum_{k=0}^2 (T_{a-k} g_k) h_{i \rightarrow i-1}^{(k)}$ gives rise to a differential symmetry breaking operator, to be denoted by D . What remains to prove is that D is a nonzero scalar multiple of

$\tilde{\mathbb{C}}_{\lambda,\nu}^{i,i-1} = \tilde{\mathcal{D}}_{\lambda-i,\nu-\lambda}^{i \rightarrow i-1}$ defined in (2.29) in the flat coordinates. We set
(10.2)

$$P := \begin{cases} \mathcal{D}_a^{\lambda-\frac{n-1}{2}} \iota_{\frac{\partial}{\partial x_n}} & \text{if } i = n, \\ \iota_{\frac{\partial}{\partial x_n}} & \text{if } \lambda = \nu, \\ -\mathcal{D}_{a-2}^{\lambda-\frac{n-3}{2}} d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} - \gamma\left(\lambda - \frac{n-1}{2}, a\right) \Pi_{n-1} \mathcal{D}_{a-1}^{\lambda-\frac{n-3}{2}} d_{\mathbb{R}^n}^* + \frac{\lambda-n+i}{2} \mathcal{D}_a^{\lambda-\frac{n-1}{2}} \iota_{\frac{\partial}{\partial x_n}} & \text{otherwise.} \end{cases}$$

We shall verify

- $\text{Symb}(P) = \begin{cases} (T_a g_0) h_{i \rightarrow i-1}^{(0)} & i = n \text{ or } \lambda = \nu, \\ e^{-\frac{\pi\sqrt{-1}(a-2)}{2}} \sum_{k=0}^2 (T_{a-k} g_k) h_{i \rightarrow i-1}^{(k)} & \text{otherwise.} \end{cases}$
- $\tilde{\mathbb{C}}_{\lambda,\nu}^{i,i-1} = \text{Rest}_{x_n=0} \circ P.$

By the general theory of the F-method (Fact 3.3), Theorem 2.9 in the case $j = i-1$ follows from these two statements. The second statement is clear from the identity $\text{Rest}_{x_n=0} \circ \Pi_{n-1} = \text{Rest}_{x_n=0}$ (see (8.26)) and the definition (2.29) of the renormalized operator $\tilde{\mathbb{C}}_{\lambda,\nu}^{i,i-1}$. The first statement in the case $i = n$ or $\lambda = \nu$ follows directly from the formula for the symbol map given in Lemma 9.1 and Proposition 8.24 (2). Thus the rest of this section will be devoted to a proof of the first statement in the case $i \neq n$ and $\lambda \neq \nu$ (see Lemma 10.2), which requires some few computations.

Let $A, B, C \in \mathbb{C}$, and we set

$$\begin{aligned} g_2(t) &= \tilde{C}_{a-2}^\mu \left(e^{\frac{\pi\sqrt{-1}}{2}t} \right), \\ g_1(t) &= e^{-\frac{\pi\sqrt{-1}}{2}t} A \tilde{C}_{a-1}^\mu \left(e^{\frac{\pi\sqrt{-1}}{2}t} \right), \\ g_0(t) &= e^{-\frac{\pi\sqrt{-1}}{2}t} B t \tilde{C}_{a-1}^\mu \left(e^{\frac{\pi\sqrt{-1}}{2}t} \right) + C \tilde{C}_{a-2}^\mu \left(e^{\frac{\pi\sqrt{-1}}{2}t} \right). \end{aligned}$$

Lemma 10.1. *Let $a \in \mathbb{N}$ and $\mu \in \mathbb{C}$. We set*

$$\begin{aligned} D_1 &:= \left(-d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* + \left(C - \frac{i-1}{n-1} \right) \Delta_{\mathbb{R}^{n-1}} \right) \iota_{\frac{\partial}{\partial x_n}}, \\ D_2 &:= -A \Pi_{n-1} \circ d_{\mathbb{R}^n}^* + (-A + B) \frac{\partial}{\partial x_n} \iota_{\frac{\partial}{\partial x_n}}. \end{aligned}$$

Then the symbol of the differential operator

$$\mathcal{D}_{a-2}^\mu D_1 + \mathcal{D}_{a-1}^\mu D_2 : \mathcal{E}^i(\mathbb{R}^n) \longrightarrow \mathcal{E}^{i-1}(\mathbb{R}^n)$$

is given by

$$\text{Symb}(\mathcal{D}_{a-2}^\mu D_1 + \mathcal{D}_{a-1}^\mu D_2) = e^{-\frac{\pi\sqrt{-1}}{2}(a-2)} \sum_{k=0}^2 (T_{a-k} g_k) h_{i \rightarrow i-1}^{(k)}.$$

Proof. We first claim the following equalities:

$$\begin{aligned} \text{Symb}(\mathcal{D}_{a-2}^\mu) &= e^{-\frac{\pi\sqrt{-1}(a-2)}{2}} T_{a-2} g_2. \\ \text{Symb}(A\mathcal{D}_{a-1}^\mu) &= e^{-\frac{\pi\sqrt{-1}(a-2)}{2}} T_{a-1} g_1. \\ \text{Symb}\left(B\mathcal{D}_{a-1}^\mu \frac{\partial}{\partial x_n} + C\mathcal{D}_{a-2}^\mu \Delta_{\mathbb{R}^{n-1}}\right) &= e^{-\frac{\pi\sqrt{-1}(a-2)}{2}} T_a g_0. \end{aligned}$$

The first two follow from Lemma 9.1. For the third equality we note that

$$T_a g_0 = e^{-\frac{\pi\sqrt{-1}}{2}} B \zeta_n T_{a-1} \left(\tilde{C}_{a-1}^\mu \left(e^{\frac{\pi\sqrt{-1}}{2}} \cdot \right) \right) + C Q_{n-1}(\zeta') T_{a-2} \left(\tilde{C}_{a-2}^\mu \left(e^{\frac{\pi\sqrt{-1}}{2}} \cdot \right) \right)$$

by Lemma 6.27 (1) and (2).

Combining the above formulæ with Proposition 8.24 (4), (3), and (2), respectively, we get

$$\begin{aligned} &\text{Symb}(-\mathcal{D}_{a-2}^\mu \left(d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* + \frac{i-1}{n-1} \Delta_{\mathbb{R}^{n-1}} \right) \iota_{\frac{\partial}{\partial x_n}} + A\mathcal{D}_{a-1}^\mu (-\Pi_{n-1} d_{\mathbb{R}^n}^* - \frac{\partial}{\partial x_n} \iota_{\frac{\partial}{\partial x_n}}) \\ &+ (B\mathcal{D}_{a-1}^\mu \frac{\partial}{\partial x_n} + C\mathcal{D}_{a-2}^\mu \Delta_{\mathbb{R}^{n-1}}) \iota_{\frac{\partial}{\partial x_n}}) \\ &= e^{-\frac{\pi\sqrt{-1}(a-2)}{2}} \left((T_{a-2} g_2) h_{i \rightarrow i-1}^{(2)} + (T_{a-1} g_1) h_{i \rightarrow i-1}^{(1)} + (T_a g_0) h_{i \rightarrow i-1}^{(0)} \right). \end{aligned}$$

A simple computation shows that the left-hand side is equal to

$$\text{Symb}(\mathcal{D}_{a-2}^\mu D_1 + \mathcal{D}_{a-1}^\mu D_2).$$

Hence Lemma 10.1 is proved. \square

We put

$$A := \gamma\left(\lambda - \frac{n-1}{2}, a\right), \quad B := \left(1 + \frac{\lambda - n + i}{a}\right) A, \quad C := \frac{\lambda - n + i}{a} + \frac{i-1}{n-1}.$$

Lemma 10.2. *Let $a := \nu - \lambda \in \mathbb{N}_+$ and $i \neq n$. Suppose $g_0(t), g_1(t)$, and $g_2(t)$ are given by the above A, B, C with $\mu = \lambda - \frac{n-3}{2}$. Then the matrix-valued differential operator P given in (10.2) satisfies*

$$\text{Symb}(P) = e^{-\frac{\pi\sqrt{-1}(a-2)}{2}} \sum_{k=0}^2 (T_{a-k} g_k) h_{i \rightarrow i-1}^{(k)}.$$

Proof. With the above constants A , B , and C , the differential operators D_1 and D_2 in Lemma 10.1 amount to

$$\begin{aligned} D_1 &= \left(-d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* + \frac{\lambda - n + i}{a} \Delta_{\mathbb{R}^{n-1}} \right) \iota_{\frac{\partial}{\partial x_n}}, \\ D_2 &= \gamma \left(-\Pi_{n-1} \circ d_{\mathbb{R}^n}^* + \frac{\lambda - n + i}{a} \frac{\partial}{\partial x_n} \iota_{\frac{\partial}{\partial x_n}} \right). \end{aligned}$$

Therefore

$$\begin{aligned} \mathcal{D}_{a-2}^{\lambda-\frac{n-3}{2}} D_1 + \mathcal{D}_{a-1}^{\lambda-\frac{n-3}{2}} D_2 &= -\mathcal{D}_{a-2}^{\lambda-\frac{n-3}{2}} d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} - \gamma \left(\lambda - \frac{n-1}{2}, a \right) \mathcal{D}_{a-1}^{\lambda-\frac{n-3}{2}} \Pi_{n-1} d_{\mathbb{R}^n}^* \\ &\quad + \frac{\lambda - n + i}{a} \left(\mathcal{D}_{a-1}^{\lambda-\frac{n-3}{2}} \Delta_{\mathbb{R}^{n-1}} + \gamma \left(\lambda - \frac{n-1}{2}, a \right) \mathcal{D}_{a-1}^{\lambda-\frac{n-3}{2}} \frac{\partial}{\partial x_n} \right) \iota_{\frac{\partial}{\partial x_n}}. \end{aligned}$$

Applying the three-term relation (9.6) of the scalar-valued differential operators \mathcal{D}_ℓ^μ , it amounts to

$$-\mathcal{D}_{a-2}^{\lambda-\frac{n-3}{2}} d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} - \gamma \left(\lambda - \frac{n-1}{2}, a \right) \Pi_{n-1} \mathcal{D}_{a-1}^{\lambda-\frac{n-3}{2}} d_{\mathbb{R}^n}^* + \frac{\lambda - n + i}{2} \mathcal{D}_a^{\lambda-\frac{n-1}{2}} \iota_{\frac{\partial}{\partial x_n}}.$$

Hence, Lemma 10.1 implies the statement of Lemma 10.2. \square

Thus we have completed the proof of Theorem 2.9 in the case $j = i - 1$.

10.2. Proof of Theorem 2.9 in the case $j = i + 1$. In this section, we give a proof of Theorem 2.9 in the case $j = i + 1$. Suppose we are in Cases 4 or 4' in Theorem 2.9, namely,

Case 4. $1 \leq i \leq n - 2$, $(\lambda, \nu) = (i, i + 1)$ and $\beta \equiv \alpha + 1 \pmod{2}$,

Case 4'. $i = 0$, $\lambda \in -\mathbb{N}$, $\nu = 1$ and $\beta \equiv \alpha + \lambda + 1 \pmod{2}$.

Then we have from Theorem 7.3

$$\text{Sol}(\mathbf{n}_+; \sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(i+1)}) = \begin{cases} \mathbb{C} \left(T_{-\lambda} \tilde{C}_{-\lambda}^{\lambda-\frac{n-1}{2}} \left(e^{\frac{\pi\sqrt{-1}}{2}} \cdot \right) \right) h_{i \rightarrow i+1}^{(1)} & \text{in Case 4'}, \\ \mathbb{C} h_{i \rightarrow i+1}^{(1)} & \text{in Case 4.} \end{cases}$$

We define a differential operator $Q: \mathcal{E}^i(\mathbb{R}^n) \longrightarrow \mathcal{E}^{i+1}(\mathbb{R}^n)$ by the formula

$$Q := \begin{cases} e^{-\frac{\pi\sqrt{-1}\lambda}{2}} \Pi_{n-1} \circ \mathcal{D}_{-\lambda}^{\lambda-i-\frac{n-1}{2}} d_{\mathbb{R}^n} & \text{if } i = 0, \\ \Pi_{n-1} \circ d_{\mathbb{R}^n}, & \text{if } 1 \leq i \leq n - 2. \end{cases}$$

We shall verify the following claims in both Case 4 and Case 4':

- $\text{Symb}(Q)$ is a generator of $\text{Sol}(\mathbf{n}_+; \sigma_{\lambda, \alpha}^{(i)}, \tau_{\nu, \beta}^{(i+1)})$.
- $\text{Rest}_{x_n=0} \circ Q: \mathcal{E}^i(\mathbb{R}^n) \longrightarrow \mathcal{E}^{i+1}(\mathbb{R}^{n-1})$ coincides with $\tilde{\mathbb{C}}_{\lambda, \nu}^{i, i+1}$.

By the general theory of the F-method (Fact 3.3), Theorem 2.9 in the case $j = i + 1$ follows from these two claims. The first claim follows from the computation of the symbol in Lemma 9.1 and Proposition 8.24 (8). Now, by use of the identity $\text{Rest}_{x_n=0} \circ \Pi_{n-1} = \text{Rest}_{x_n=0}$ (see (8.26)) and the definition (2.30) of $\tilde{\mathbb{C}}_{\lambda,\nu}^{i,i+1} (= \tilde{\mathcal{D}}_{\lambda-i,\nu-\lambda}^{i \rightarrow i+1})$, we obtain $\tilde{\mathbb{C}}_{\lambda,\nu}^{i,i+1} = \text{Rest}_{x_n=0} \circ Q$. Thus we have completed the proof of Theorem 2.9 in the case $j = i + 1$.

10.3. Application of the duality theorem for symmetry breaking operators.

In the following two Sections 10.4 and 10.5, we shall give a proof of Theorem 2.9 in the cases $j = i$ and $i - 2$ by applying the duality theorem for symmetry breaking operators (Theorem 2.7), instead of solving the F-system. We shall see that the cases $j = i$ and $i - 2$ are derived from the cases $\tilde{j} = \tilde{i} - 1$ and $\tilde{i} + 1$, for which the proof was completed in Sections 10.1 and 10.2, respectively.

In this section we give a set-up for the duality theorem. We put

$$\tilde{i} := n - i, \quad \tilde{j} := n - 1 - j.$$

First we examine a geometric meaning of the proof of Lemma 2.2 and Theorem 2.7. Let χ_{--} be the one-dimensional representation of G as defined in (2.9). Then the proof of Lemma 2.2 shows that the Hodge star operator on $\mathcal{E}^i(\mathbb{R}^n)$ induces the G -isomorphism $I(i, \lambda)_\alpha \simeq I(\tilde{i}, \lambda)_\alpha \otimes \chi_{--}$ in the flat picture (see (2.8)) as below:

$$(10.3) \quad \begin{array}{ccc} \mathcal{E}^i(\mathbb{R}^n) & \xrightarrow{*_{\mathbb{R}^n}} & \mathcal{E}^{\tilde{i}}(\mathbb{R}^n) \simeq \mathcal{E}^{\tilde{i}}(\mathbb{R}^n) \otimes \mathbb{C} \\ \iota_\lambda^{(i)} \uparrow & & \uparrow \iota_\lambda^{(\tilde{i})} \\ I(i, \lambda)_\alpha & \longrightarrow & I(\tilde{i}, \lambda)_\alpha \otimes \chi_{--}. \end{array}$$

We recall the proof of Theorem 2.7 is based on the G - and G' -isomorphisms

$$\begin{aligned} I(i, \lambda)_\alpha &\simeq I(\tilde{i}, \lambda)_\alpha \otimes \chi_{--}, \\ J(j, \nu)_\beta &\simeq J(\tilde{j}, \nu)_\beta \otimes \chi_{--}|_{G'}, \end{aligned}$$

which induce the duality of symmetry breaking operators

$$\text{Diff}_{G'}(I(i, \lambda)_\alpha, J(j, \nu)_\beta) \simeq \text{Diff}_{G'}(I(\tilde{i}, \lambda)_\alpha \otimes \chi_{--}, J(\tilde{j}, \nu)_\beta \otimes \chi_{--}|_{G'}), \quad T \mapsto \tilde{T} \otimes \text{id}.$$

In the flat picture, this isomorphism is realized by (10.3) in the following key diagram:

$$\begin{array}{ccccccc} \mathcal{E}^i(\mathbb{R}^n) & \longleftarrow & I(i, \lambda)_\alpha & \xrightarrow{T} & J(j, \nu)_\beta & \hookrightarrow & \mathcal{E}^j(\mathbb{R}^{n-1}) \\ \uparrow *_{\mathbb{R}^n} & & \parallel & & \parallel & & \uparrow *_{\mathbb{R}^{n-1}} \\ \mathcal{E}^{\tilde{i}}(\mathbb{R}^n) & \longleftarrow & I(\tilde{i}, \lambda)_\alpha \otimes \chi_{--} & \xrightarrow{\tilde{T} \otimes \text{id}} & J(\tilde{j}, \nu)_\beta \otimes \chi_{--} & \hookrightarrow & \mathcal{E}^{\tilde{j}}(\mathbb{R}^{n-1}) \end{array}$$

We note that the 6-tuple $(i, j, \lambda, \nu, \alpha, \beta)$ for $j = i$ belongs to Case 3 in Theorem 2.8 (which we shall consider in this section) if and only if $(\tilde{i}, \tilde{i} - 1, \lambda, \nu, \alpha, \beta)$ belongs to Case 2 in Theorem 2.8 (which was treated in Section 10.1). Likewise $(i, j, \lambda, \nu, \alpha, \beta)$ for $j = i - 2$ belongs to Cases 1 and 1' in Theorem 2.8 if and only if $(\tilde{i}, \tilde{i} + 1, \lambda, \nu, \alpha, \beta)$ belongs to Cases 4 and 4' in Theorem 2.8 (which were treated in Section 10.2).

In view of the above geometric interpretation of the duality theorem (Theorem 2.7), Theorem 2.9 in the case $j = i$ and $i - 2$ is deduced from the following identities

$$(10.4) \quad \tilde{\mathbb{C}}_{\lambda, \nu}^{i, i} = (-1)^{n-1} *_{\mathbb{R}^{n-1}} \circ \tilde{\mathbb{C}}_{\lambda, \nu}^{\tilde{i}, \tilde{i}-1} \circ (*_{\mathbb{R}^n})^{-1},$$

$$(10.5) \quad \tilde{\mathbb{C}}_{\lambda, \nu}^{i, i-2} = (-1)^{n-1} *_{\mathbb{R}^{n-1}} \circ \tilde{\mathbb{C}}_{\lambda, \nu}^{\tilde{i}, \tilde{i}+1} \circ (*_{\mathbb{R}^n})^{-1},$$

in the flat picture, which will be treated in Propositions 10.3 and 10.4, respectively, in the next two sections.

10.4. Proof of Theorem 2.9 in the case $j = i$. In this section, we prove the duality (10.2) as well as the equality (2.23) = (2.24) for the two expressions of $\mathbb{C}_{\lambda, \nu}^{i, i}$ (or equivalently, (1.6) = (1.7) for $\mathcal{D}_{u, a}^{i \rightarrow i}$), and complete the proof of Theorem 2.9 in the case $j = i$.

Proposition 10.3. *Let $0 \leq i \leq n - 1$, and $(\lambda, \nu) \in \mathbb{C}^2$ with $\nu - \lambda \in \mathbb{N}$. We consider a matrix-valued differential operator*

$$(10.6) \quad (-1)^{n-1} *_{\mathbb{R}^{n-1}} \circ \tilde{\mathbb{C}}_{\lambda, \nu}^{\tilde{i}, \tilde{i}-1} \circ (*_{\mathbb{R}^n})^{-1} : \mathcal{E}^i(\mathbb{R}^n) \longrightarrow \mathcal{E}^i(\mathbb{R}^{n-1}),$$

where $\tilde{i} := n - i$. Then (10.6) and the two expressions (2.23), (2.24) of $\mathbb{C}_{\lambda, \nu}^{i, i}$ are equal to each other. Moreover, we have the following identity for the renormalized symmetry breaking operators (see (2.29) for the definition)

$$(10.7) \quad \tilde{\mathbb{C}}_{\lambda, \nu}^{i, i} = (-1)^{n-1} *_{\mathbb{R}^{n-1}} \circ \tilde{\mathbb{C}}_{\lambda, \nu}^{\tilde{i}, \tilde{i}-1} \circ (*_{\mathbb{R}^n})^{-1}.$$

Proof. We recall the notation from (2.21) that $\tilde{\mathbb{C}}_{\lambda, \nu} = \text{Rest}_{x_n=0} \circ \mathcal{D}_{\nu-\lambda}^{\lambda-\frac{n-1}{2}}$. By (2.25) or by (1.4), we have

$$\begin{aligned} \tilde{\mathbb{C}}_{\lambda, \nu}^{\tilde{i}, \tilde{i}-1} &= \mathcal{D}_{\lambda-\tilde{i}, \nu-\lambda}^{\tilde{i} \rightarrow \tilde{i}-1} \\ &= \text{Rest}_{x_n=0} \circ \left(-\mathcal{D}_{\nu-\lambda-2}^{\lambda-\frac{n-3}{2}} d_{\mathbb{R}^n}^* d_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}} - \gamma \left(\lambda - \frac{n-1}{2}, \nu - \lambda \right) \mathcal{D}_{\nu-\lambda-1}^{\lambda-\frac{n-3}{2}} d_{\mathbb{R}^n}^* + \frac{1}{2} (\lambda - i) \mathcal{D}_{\nu-\lambda}^{\lambda-\frac{n-1}{2}} \iota_{\frac{\partial}{\partial x_n}} \right). \end{aligned}$$

Applying the formulæ for $*_{\mathbb{R}^{n-1}} \circ \text{Rest}_{x_n=0} \circ T \circ (*_{\mathbb{R}^n})^{-1}$ given in Lemma 8.20 ($T = d_{\mathbb{R}^n}^*$ and $\iota_{\frac{\partial}{\partial x_n}}$) and in Lemma 8.22 ($T = d_{\mathbb{R}^n}^* d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}}$), we obtain

$$\begin{aligned} & (-1)^{n-1} *_{\mathbb{R}^{n-1}} \circ \tilde{\mathbb{C}}_{\lambda, \nu}^{\tilde{i}, \tilde{i}-1} \circ (*_{\mathbb{R}^n})^{-1} \\ &= -d_{\mathbb{R}^{n-1}}^* d_{\mathbb{R}^{n-1}} \text{Rest}_{x_n=0} \circ \mathcal{D}_{\nu-\lambda-2}^{\lambda-\frac{n-3}{2}} \\ &\quad + \text{Rest}_{x_n=0} \circ \left(\gamma \left(\lambda - \frac{n-1}{2}, \nu - \lambda \right) \mathcal{D}_{\nu-\lambda-1}^{\lambda-\frac{n-3}{2}} \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n} + \frac{1}{2} (\lambda - i) \mathcal{D}_{\nu-\lambda}^{\lambda-\frac{n-1}{2}} \right) \\ &= -d_{\mathbb{R}^{n-1}}^* d_{\mathbb{R}^{n-1}} \tilde{\mathbb{C}}_{\lambda+1, \nu-1} + \gamma \left(\lambda - \frac{n-1}{2}, \nu - \lambda \right) \tilde{\mathbb{C}}_{\lambda+1, \nu} \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n} + \frac{\lambda - i}{2} \tilde{\mathbb{C}}_{\lambda, \nu}. \end{aligned}$$

Hence we have proved the equality (10.6) = (2.24).

On the other hand, we have proved in Proposition 9.9 that $\mathbb{C}_{\lambda, \nu}^{\tilde{i}, \tilde{i}-1}$ is equal to

$$\text{Rest}_{x_n=0} \circ \left(-\mathcal{D}_{\nu-\lambda-2}^{\lambda-\frac{n-3}{2}} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n} + \frac{1}{2} (\nu - i) \mathcal{D}_{\nu-\lambda}^{\lambda-\frac{n-1}{2}} \iota_{\frac{\partial}{\partial x_n}} - \gamma \left(\lambda - \frac{n}{2}, \nu - \lambda \right) \mathcal{D}_{\nu-\lambda-1}^{\lambda-\frac{n-1}{2}} \left(d_{\mathbb{R}^n}^* + \frac{\partial}{\partial x_n} \iota_{\frac{\partial}{\partial x_n}} \right) \right).$$

Applying Lemma 8.20 to $T = -d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n}$, and $d_{\mathbb{R}^n}^* + \frac{\partial}{\partial x_n} \iota_{\frac{\partial}{\partial x_n}}$, we have

$$\begin{aligned} & (-1)^{n-1} *_{\mathbb{R}^{n-1}} \circ \tilde{\mathbb{C}}_{\lambda, \nu}^{\tilde{i}, \tilde{i}-1} \circ (*_{\mathbb{R}^n})^{-1} \\ &= \text{Rest}_{x_n=0} \circ \left(\mathcal{D}_{\nu-\lambda-2}^{\lambda-\frac{n-3}{2}} d_{\mathbb{R}^n}^* d_{\mathbb{R}^n}^* + \frac{1}{2} (\nu - i) \mathcal{D}_{\nu-\lambda}^{\lambda-\frac{n-1}{2}} - \gamma \left(\lambda - \frac{n}{2}, \nu - \lambda \right) \mathcal{D}_{\nu-\lambda-1}^{\lambda-\frac{n-1}{2}} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} \right) \\ &= \tilde{\mathbb{C}}_{\lambda+1, \nu-1} d_{\mathbb{R}^n}^* d_{\mathbb{R}^n}^* + \frac{1}{2} (\nu - i) \tilde{\mathbb{C}}_{\lambda, \nu} - \gamma \left(\lambda - \frac{n}{2}, \nu - \lambda \right) \tilde{\mathbb{C}}_{\lambda, \nu-1}, \end{aligned}$$

which is equal to the formula (2.23). Thus we have shown the equalities: (2.23) = (2.24) = (10.4).

Finally, let us prove the identity (10.7). We have already shown (10.7) when $\lambda \neq \nu$ and $i \neq 0$ (i.e. $\tilde{i} \neq n$) because $\tilde{\mathbb{C}}_{\lambda, \nu}^{i, i} = \mathbb{C}_{\lambda, \nu}^{i, i}$ and $\tilde{\mathbb{C}}_{\lambda, \nu}^{\tilde{i}, \tilde{i}-1} = \mathbb{C}_{\lambda, \nu}^{\tilde{i}, \tilde{i}-1}$ in this case. For $\lambda = \nu$ or $i = 0$ (i.e. $\tilde{i} = n$), the identity (10.7) is an immediate consequence of the definition (2.29) and Lemma 8.20 with $T = \iota_{\frac{\partial}{\partial x_n}}$. Thus the proof of the proposition is completed. \square

10.5. Proof of Theorem 2.9 in the case $j = i - 2$. In this section, we prove Theorem 2.9 in the remaining case, namely, $j = i - 2$. We keep the notation $(\tilde{i}, \tilde{j}) = (n - i, n - 1 - j)$, and assume $j = i - 2$ in this section. Then Cases 1 (resp. 1') and 4 (resp. 4') in Theorem 2.8 are dual to each other, namely,

Case 4: $\tilde{j} = \tilde{i} + 1$, $1 \leq \tilde{i} \leq n - 2$, $(\lambda, \nu) = (\tilde{i}, \tilde{i} + 1)$, $\beta \equiv \alpha + 1 \pmod{2}$,

Case 4': $(\tilde{i}, \tilde{j}) = (0, 1)$, $\lambda \in -\mathbb{N}$, $\nu = 1$, $\beta \equiv \alpha + \lambda + 1 \pmod{2}$,

are equivalent to

Case 1: $j = i - 2$, $2 \leq i \leq n - 1$, $(\lambda, \nu) = (n - i, n - i + 1)$, $\beta \equiv \alpha + 1 \pmod{2}$,

Case 1': $(i, j) = (n, n - 2)$, $\lambda \in -\mathbb{N}$, $\nu = 1$, $\beta \equiv \alpha + \lambda + 1 \pmod{2}$,

respectively.

By the duality (Theorem 2.7) and the proof of Theorem 2.9 in the case $\tilde{j} = \tilde{i} + 1$, Theorem 2.9 in the case $j = i - 2$ is deduced from the following proposition.

Proposition 10.4. *We have the identity (10.5), namely,*

$$\begin{aligned}\tilde{\mathbb{C}}_{n-i, n-i+1}^{i, i-2} &= (-1)^{n-1} *_{\mathbb{R}^{n-1}} \circ \tilde{\mathbb{C}}_{i, i+1}^{\tilde{i}, \tilde{i}+1} \circ (*_{\mathbb{R}^n})^{-1} \quad \text{in Case 1,} \\ \tilde{\mathbb{C}}_{\lambda, 1}^{n, n-2} &= (-1)^{n-1} *_{\mathbb{R}^{n-1}} \circ \tilde{\mathbb{C}}_{\lambda, 1}^{0, 1} \circ (*_{\mathbb{R}^n})^{-1} \quad \text{in Case 1'.}\end{aligned}$$

Proof. We recall from (2.30) that $\tilde{\mathbb{C}}_{\lambda, \nu}^{\tilde{i}, \tilde{i}+1} = \text{Rest}_{x_n=0} \circ \mathcal{D}_{\tilde{i}-\lambda}^{\lambda-\tilde{i}-\frac{n-1}{2}} d_{\mathbb{R}^n}$. It follows from Lemma 8.20 (1) and from Lemma 8.17 with $T = d_{\mathbb{R}^n}$ that

$$(-1)^{n-1} *_{\mathbb{R}^{n-1}} \circ \text{Rest}_{x_n=0} \circ d_{\mathbb{R}^n} \circ (*_{\mathbb{R}^n})^{-1} = \text{Rest}_{x_n=0} \circ \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n}^*.$$

Hence we have

$$(10.8) \quad (-1)^{n-1} *_{\mathbb{R}^{n-1}} \circ \tilde{\mathbb{C}}_{\lambda, \nu}^{\tilde{i}, \tilde{i}+1} \circ (*_{\mathbb{R}^n})^{-1} = \text{Rest}_{x_n=0} \circ \mathcal{D}_{\tilde{i}-\lambda}^{\lambda-\tilde{i}-\frac{n-1}{2}} \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n}^*.$$

In Case 1, $\lambda = \tilde{i}$ and therefore (10.8) amounts to $\text{Rest}_{x_n=0} \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n}^* = \tilde{\mathbb{C}}_{n-i, n-i+1}^{i, i-2}$.

In Case 1', $i = n$, $\tilde{i} = 0$, and therefore (10.8) amounts to $\text{Rest}_{x_n=0} \circ \mathcal{D}_{-\lambda}^{\lambda-\frac{n-1}{2}} \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n}^*$ which is equal to $\tilde{\mathbb{C}}_{\lambda, 1}^{n, n-2}$. \square

Hence the proof of Theorem 2.9 is completed.

11. SOLUTIONS TO PROBLEMS A AND B FOR (S^n, S^{n-1})

In this chapter, we complete the proof of Theorem 1.1 and Theorems 1.5–1.8, which solve Problems A and B of conformal geometry for the model space $(X, Y) = (S^n, S^{n-1})$, respectively.

11.1. Problems A and B for conformal transformation group $\text{Conf}(X; Y)$.

We begin with the general setting where (X, g_X) is a pseudo-Riemannian manifold of dimension n , and Y is a submanifold of dimension m such that the metric tensor g_X is nondegenerate when restricted to Y . We define $\text{Conf}(X; Y)$ as a subgroup of the full conformal group $\text{Conf}(X) := \{\varphi: X \rightarrow X \text{ is a conformal diffeomorphism}\}$ by

$$(11.1) \quad \text{Conf}(X; Y) := \{\varphi \in \text{Conf}(X) : \varphi(Y) = Y\}.$$

Then $\mathcal{E}^i(X)_{u,\delta}$ is a $\text{Conf}(X)$ -module for $0 \leq i \leq n$, $u \in \mathbb{C}$, $\delta \in \mathbb{Z}/2\mathbb{Z}$, and $\mathcal{E}^j(Y)_{v,\varepsilon}$ is a $\text{Conf}(X; Y)$ -module for $0 \leq j \leq m$, $v \in \mathbb{C}$, $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. The group $\text{Conf}(X; Y)$ is the largest effective group for Problems A and B on differential symmetry breaking operators from $\mathcal{E}^i(X)_{u,\delta}$ to $\mathcal{E}^j(Y)_{v,\varepsilon}$.

The first reduction is the duality theorem for symmetry breaking operators. We recall from Proposition 8.3 that the Hodge star operator $*_X$ carrying i -forms to $(n-i)$ -forms is a conformally equivariant operator for X , and $*_Y$ carrying j -forms to $(m-j)$ -forms is a conformally equivariant for Y . Then, a solution to Problem A (or Problem B) for i -forms on X and j -forms on the submanifold Y , to be denoted by the (i, j) case, leads us to solutions for $(i, m-j)$, $(n-i, j)$, and $(n-i, m-j)$ cases via the following natural bijections:

$$(11.2) \quad \begin{array}{ccc} \text{Diff}_{\text{Conf}(X; Y)}(\mathcal{E}^i(X)_{u,0}, \mathcal{E}^j(Y)_{v,0}) & \xlongequal{\quad} & \text{Diff}_{\text{Conf}(X; Y)}(\mathcal{E}^i(X)_{u,0}, \mathcal{E}^{m-j}(Y)_{v-m+2j,1}) \\ \Big\downarrow & & \Big\downarrow \\ \text{Diff}_{\text{Conf}(X; Y)}(\mathcal{E}^{n-i}(X)_{u-n+2i,1}, \mathcal{E}^j(Y)_{v,0}) & \xlongequal{\quad} & \text{Diff}_{\text{Conf}(X; Y)}(\mathcal{E}^{n-i}(X)_{u-n+2i,1}, \mathcal{E}^{m-j}(Y)_{v-m+2j,1}), \end{array}$$

given by

$$(11.3) \quad \begin{array}{ccc} D & \xrightarrow{\quad} & *_Y \circ D \\ \downarrow & & \downarrow \\ D \circ *_X & \xrightarrow{\quad} & *_Y \circ D \circ *_X. \end{array}$$

In other words, a solution to Problem A (or Problem B) for a fixed $(\delta, \varepsilon) \in (\mathbb{Z}/2\mathbb{Z})^2$ yields solutions to Problem A (or Problem B, respectively) for the other three cases of $(\delta, \varepsilon) \in (\mathbb{Z}/2\mathbb{Z})^2$.

11.2. Model space $(X, Y) = (S^n, S^{n-1})$. From now we consider the model space $(X, Y) = (S^n, S^{n-1})$. We shall see that Problems A and B are deduced from the problems on symmetry breaking operators between principal series representations of $G = O(n+1, 1)$ and $G' = O(n, 1)$ which were proved in Theorems 2.8 and 2.9, respectively. For this, we first clarify small differences such as disconnected components and coverings between the groups G and $\text{Conf}(X)$, and also between G' and $\text{Conf}(X; Y)$.

We recall from Section 2.1 that the natural action of $G = O(n+1, 1)$ on the light cone $\Xi \subset \mathbb{R}^{n+1,1}$ induces a conformal action on the standard Riemann sphere S^n via the isomorphism $S^n \simeq \Xi/\mathbb{R}^\times$. Conversely, it is well-known that any conformal transformation of the standard sphere $X = S^n$ is obtained in this manner if $n \geq 2$, and thus we have a natural isomorphism:

$$(11.4) \quad \text{Conf}(X) \simeq O(n+1, 1)/\{\pm I_{n+2}\}.$$

Let us compute $\text{Conf}(X; Y)$ for $(X, Y) = (S^n, S^{n-1})$. We realize $Y = S^{n-1}$ as a submanifold $\{(x_0, \dots, x_{n-1}, x_n) \in S^n : x_n = 0\}$ of $X = S^n$ as before.

Lemma 11.1. *Via the isomorphism (11.4), we have*

$$\text{Conf}(X; Y) \simeq (O(n, 1) \times O(1)) / \{\pm I_{n+2}\}.$$

Proof. Suppose $g = (g_{ij})_{0 \leq i, j \leq n+1} \in O(n+1, 1)$ leaves $Y = S^{n-1}$ invariant. This means that $\sum_{j=0}^{n+1} g_{nj} \xi_j = 0$ for all $\xi = (\xi_0, \dots, \xi_{n+1}) \in \Xi$ with $\xi_n = 0$, which implies $g_{nj} = 0$ for all $j \neq n$. In turn, $g_{in} = 0$ for all $i \neq n$ and $g_{nn} = \pm 1$ because $g \in O(n+1, 1)$. Hence we have shown $g \in O(n, 1) \times O(1)$. Conversely, any element of $O(n, 1) \times O(1)$ clearly leaves S^{n-1} invariant. Thus the lemma is proved. \square

The above lemma says that the group $\text{Conf}(X; Y)$ is the quotient of the direct product group of $G' = O(n, 1)$ and $O(1)$, however, we do not have to consider the second factor $O(1)$ in solving Problems A and B. In order to state this claim precisely, we write

$$\delta \cdot i := \begin{cases} i & \text{if } \delta \equiv 0 \\ n-i & \text{if } \delta \equiv 1, \end{cases} \quad \varepsilon \cdot j := \begin{cases} j & \text{if } \varepsilon \equiv 0 \\ n-1-j & \text{if } \varepsilon \equiv 1, \end{cases}$$

for $\delta, \varepsilon \in \mathbb{Z}/2\mathbb{Z}$ and $0 \leq i \leq n, 0 \leq j \leq n-1$. We recall that $I(i, \lambda)_\alpha$ is a principal series representation with parameter $\lambda \in \mathbb{C}$ and $\alpha \in \mathbb{Z}/2\mathbb{Z}$ of $G = O(n+1, 1)$, and $J(j, \nu)_\beta$ is that of $G' = O(n, 1)$. Then we have

Lemma 11.2. *For $(X, Y) = (S^n, S^{n-1})$, we have a natural isomorphism:*

$$(11.5) \quad \text{Hom}_{\text{Conf}(X; Y)}(\mathcal{E}^i(X)_{u, \delta}, \mathcal{E}^j(Y)_{v, \varepsilon}) \simeq \text{Hom}_{O(n, 1)}(I(\delta \cdot i, u+i)_{\delta \cdot i}, J(\varepsilon \cdot j, v+j)_{\varepsilon \cdot j}).$$

Here the subscripts $\delta \cdot i$ and $\varepsilon \cdot j$ are regarded as elements in $\mathbb{Z}/2\mathbb{Z}$.

Proof. For $\alpha \in \mathbb{Z}/2\mathbb{Z}$, we write $(-1)^\alpha$ for the one-dimensional representation of $O(1)$ as before, namely,

$$(-1)^\alpha = \begin{cases} \mathbb{1} & (\text{trivial representation}) & \text{if } \alpha \equiv 0, \\ \text{sgn} & (\text{signature representation}) & \text{if } \alpha \equiv 1. \end{cases}$$

Since the central element $-I_{n+2}$ of G acts on the principal series representation $I(i, \lambda)_\alpha$ as the scalar $(-1)^{i+\alpha}$, and since $-I_{n+1}$ acts on $J(j, \nu)_\beta$ as the scalar $(-1)^{j+\beta}$, we have

$$\begin{aligned} & \text{Hom}_{O(n,1) \times O(1)} (I(i, \lambda)_\alpha, J(j, \nu)_\beta \boxtimes (-1)^\gamma) \\ & \simeq \begin{cases} \text{Hom}_{O(n,1)} (I(i, \lambda)_\alpha, J(j, \nu)_\beta) & \text{if } \gamma \equiv i + j + \alpha + \beta \pmod{2}, \\ \{0\} & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand, since the second factor $O(1)$ acts trivially on the submanifold $Y = S^{n-1}$, Proposition 2.3 implies that the representation $\varpi_{v,\varepsilon}^{(j)}$ of $\text{Conf}(X; Y)$ on $\mathcal{E}^j(S^{n-1})$ is given by the outer tensor product representation of $O(n, 1) \times O(1)$ as below:

$$\varpi_{v,\varepsilon}^{(j)} \simeq J(\varepsilon \cdot j, v + j)_{\varepsilon \cdot j} \boxtimes \mathbb{1}.$$

Again by Proposition 2.3, we have an isomorphism $\varpi_{u,\delta}^{(i)} \simeq I(\delta \cdot i, u + i)_{\delta \cdot i}$ as representations of $G = O(n + 1, 1)$. Thus we conclude

$$\begin{aligned} \text{Hom}_{\text{Conf}(X; Y)} (\varpi_{u,\delta}^{(i)}, \varpi_{v,\varepsilon}^{(j)}) & \simeq \text{Hom}_{O(n,1) \times O(1)} (I(\delta \cdot i, u + i)_{\delta \cdot i}, J(\varepsilon \cdot j, v + j)_{\varepsilon \cdot j} \boxtimes \mathbb{1}) \\ & \simeq \text{Hom}_{O(n,1)} (I(\delta \cdot i, u + i)_{\delta \cdot i}, J(\varepsilon \cdot j, v + j)_{\varepsilon \cdot j}). \end{aligned}$$

Hence the lemma is proved. \square

11.3. Proof of Theorem 1.1. In this section we complete the proof of Theorem 1.1. We shall see that Theorem 1.1 (conformal geometry) is derived from Theorem 2.8 (representation theory). Actually, we only need principal series representations $I(i', \lambda)_\alpha$ and $J(j', \nu)_\beta$ with $\alpha \equiv i' \pmod{2}$ and $\beta \equiv j' \pmod{2}$ in order to classify $\text{Diff}_{G'}(\mathcal{E}^i(S^n)_{u,\delta}, \mathcal{E}^j(S^{n-1})_{v,\varepsilon})$, see Remark 2.4.

Suppose that a symmetry breaking operator $D: I(i', \lambda)_\alpha \longrightarrow J(j', \nu)_\beta$ with $\alpha \equiv i' \pmod{2}$ and $\beta \equiv j' \pmod{2}$ is given. We set

$$\tilde{i}' := n - i', \quad \tilde{j}' := n - 1 - j', \quad b := j' - i', \quad \tilde{b} := -b - 1.$$

We note $\tilde{b} = \tilde{j}' - \tilde{i}'$ and that $b \mapsto \tilde{b}$ defines a permutation of the finite set $\{-2, -1, 0, 1\}$. Then the diagram (11.3) of the double dualities induces four symmetry breaking operators $T: \mathcal{E}^i(S^n)_{u,\delta} \longrightarrow \mathcal{E}^j(S^{n-1})_{v,\varepsilon}$ with $(T, i, j, u, v, \delta, \varepsilon)$ listed in Table 11.1 below:

TABLE 11.1. Conditions for $(T, i, j, u, v, \delta, \varepsilon)$ in the double dualities

T	i	j	u	v	δ	ε
D	i'	$j' = i' + b$	$\lambda - i'$	$\nu - j' = \nu - i' - b$	0	0
$* \circ D \circ *$	\tilde{i}'	$\tilde{j}' = \tilde{i}' + \tilde{b}$	$\lambda - \tilde{i}'$	$\nu - \tilde{j}' = \nu - \tilde{i}' - \tilde{b}$	1	1
$* \circ D$	i'	$\tilde{j}' = n - i' + \tilde{b}$	$\lambda - i'$	$\nu - \tilde{j}' = \nu + i' - n - \tilde{b}$	0	1
$D \circ *$	\tilde{i}'	$j' = n - \tilde{i}' + b$	$\lambda - \tilde{i}'$	$\nu - j' = \nu + \tilde{i}' - n - b$	1	0

In the columns in Table 11.1, we give formulæ for v in two ways for later purpose. We note that b or \tilde{b} gives a relationship between i and j .

Let us translate Theorem 2.8 on symmetry breaking operators for principal series representations into those for conformal geometry via the isomorphism (11.5) by using the dictionary in Table 11.1. The resulting list is given in Table 11.2. We note that among the six cases in Theorem 2.8 (iii), Case 1 does not contribute to Problem A for $(X, Y) = (S^n, S^{n-1})$ because Proposition 5.19 (1) requires $\nu - \lambda \equiv \beta - \alpha \pmod{2}$ for $\text{Diff}_{G'}(I(i', \lambda)_\alpha, J(j', \nu)_\beta)$ not to be zero, whereas $\nu - \lambda = (n - i' + 1) - (n - i') = 1$ in Case 1 does not have the same parity with $\beta - \alpha$ if we take $\alpha \equiv i'$ and $\beta \equiv j' (= i' - 2) \pmod{2}$. Then the remaining five cases in Theorem 2.8 (iii) yield $5 \times 4 = 20$ cases according to the choice of $(\alpha, \beta) \in (\mathbb{Z}/2\mathbb{Z})^2$, which are listed in Table 11.2.

Let us explain Table 11.2 in more details. We fix a case among the five cases 1', 2, 3, 4, or 4' in Theorem 2.8 (iii), choose $(\alpha, \beta) \in (\mathbb{Z}/2\mathbb{Z})^2$, and take a nonzero $D \in \text{Hom}_{G'}(I(i', \lambda)_\alpha, J(j', \nu)_\beta)$ which is unique up to scalar multiplication. Here we assume $\alpha \equiv i'$ and $\beta \equiv j' \pmod{2}$, which was not necessary in Theorem 2.8 (iii). The operators $T = D, * \circ D, D \circ *,$ or $* \circ D \circ *$ (see (11.3)) are listed in Table 11.2 according to the choice of $(\delta, \varepsilon) \in (\mathbb{Z}/2\mathbb{Z})^2$, and the operator T gives a symmetry breaking operator $\mathcal{E}^i(S^n)_{u, \delta} \rightarrow \mathcal{E}^j(S^{n-1})_{v, \varepsilon}$ where (i, j, u, v) is determined by the formulæ $(i', j', \lambda, \nu) \mapsto (i, j, u, v)$ given by Table 11.1 for each fixed $\delta, \varepsilon \in \mathbb{Z}/2\mathbb{Z}$. This procedure transforms the classification data given in Theorem 2.8 (iii) with the additional parity condition $\alpha \equiv i'$ and $\beta \equiv j'$ into Table 11.2.

For instance, $* \circ (4) \circ *$ in Table 11.2 means the following: we begin with parameter $(i', j', \lambda, \nu, \alpha, \beta)$ belonging to Case 4 in Theorem 2.8 (iii), namely, $j' = i' + 1, 1 \leq i' \leq n - 2, (\lambda, \nu) = (i', i' + 1)$, take $D \in \text{Diff}_{O(n,1)}(I(i', \lambda)_\alpha, J(j', \nu)_\beta)$ with $\alpha \equiv i'$ and $\beta \equiv j' \pmod{2}$, and then obtain $* \circ D \circ * \in \text{Diff}_{O(n,1)}(\mathcal{E}^i(S^n)_{u, \delta}, \mathcal{E}^j(S^{n-1})_{v, \varepsilon})$ where $(i, u, \delta, j, v, \varepsilon)$ is determined by

$$\delta = \varepsilon \equiv 1 \pmod{2}, \quad i = \tilde{i}' (= n - i'), \quad j = \tilde{j}' (= n - 1 - j'), \quad u = \lambda - \tilde{i}', \quad \text{and} \quad v = \nu - \tilde{j}'.$$

A short computation shows that

$$j = i - 2, \quad 2 \leq i \leq n - 1, \quad \text{and} \quad (u, v) = (n - 2i, n - 2i + 3),$$

giving the first row of Table 11.2.

The order of differential symmetry breaking operators of D is given by $a := \nu - \lambda$. Since the Hodge star operator is of order zero as a differential operator, the operators $* \circ D \circ *$, $* \circ D$, and $D \circ *$ have the same order a . We listed also the data for a in Table 11.2. Collecting these data according to the values of j and i , we get the classification of the 6-tuples $(i, j, u, v, \delta, \varepsilon)$ for the nonvanishing of $\text{Diff}_{O(n,1)}(\mathcal{E}^i(S^n)_{u,\delta}, \mathcal{E}^j(S^{n-1})_{v,\varepsilon})$ as listed in Table 11.2, or exactly the condition in Theorem 1.1 (iii). Thus Theorem 1.1 is proved.

11.4. Proof of Theorems 1.5–1.8. Theorems 1.5, 1.6, 1.7, and 1.8 are derived from Theorem 2.9 by using Table 11.2 and by the formula $\tilde{\mathbb{C}}_{\lambda,\nu}^{i,j} = \tilde{\mathcal{D}}_{u,a}^{i \rightarrow j}$ with $a = \nu - \lambda$ and $u = \lambda - i$ (see (2.22)) and the duality results (Propositions 10.3 and 10.4).

We give a proof of Theorem 1.5 below. The other three theorems are similarly shown.

Proof of Theorem 1.5. There are two rows in Table 11.2 that deal with the case $j = i - 1$. The symmetry breaking operator T in this case is given as

$$T = \begin{cases} (2) & \text{for } (\delta, \varepsilon) = (0, 0), \\ * \circ (3) \circ * & \text{for } (\delta, \varepsilon) = (1, 1), \end{cases}$$

where $T = (2)$ means that T is proportional to $\tilde{\mathbb{C}}_{\lambda,\nu}^{i,i-1}$ in the flat picture corresponding to Case 2 of Theorem 2.9, and $T = * \circ (3) \circ *$ means that T is proportional to $*_{\mathbb{R}^{n-1}} \circ \tilde{\mathbb{C}}_{\lambda,\nu}^{n-i,n-i} \circ *_{\mathbb{R}^n}$ corresponding to Case 3 of Theorem 2.9. It follows from Proposition 10.3 that the latter equals $\pm \tilde{\mathbb{C}}_{\lambda,\nu}^{i,i-1}$. In both cases, T is proportional to $\tilde{\mathbb{C}}_{\lambda,\nu}^{i,i-1} = \tilde{\mathcal{D}}_{\lambda-i,\nu-\lambda}^{i \rightarrow i-1}$ (see (2.21)). Thus Theorem 1.5 is proved. \square

11.5. Change of coordinates in symmetry breaking operators. So far we have discussed explicit formulæ of symmetry breaking operators in the flat coordinates. This section explains how to compute explicit symmetry breaking operators in the coordinates of $(X, Y) = (S^n, S^{n-1})$ from the formulæ that we found in the flat coordinates of $(\mathbb{R}^n, \mathbb{R}^{n-1})$.

We recall from (2.5) and (2.4) that the stereographic projection and its inverse are given, respectively by

$$\begin{aligned} p: S^n \setminus \{[\xi^-]\} &\longrightarrow \mathbb{R}^n, & \omega = {}^t(\omega_0, \dots, \omega_n) &\mapsto \frac{1}{1 + \omega_0} {}^t(\omega_1, \dots, \omega_n), \\ \iota: \mathbb{R}^n &\longrightarrow S^n, & x = {}^t(x_1, \dots, x_n) &\mapsto \frac{1}{1 + Q_n(x)} {}^t(1 - Q_n(x), 2x_1, \dots, 2x_n), \end{aligned}$$

TABLE 11.2. Relation between Theorem 1.1 for $\text{Diff}_{O(n,1)}(\mathcal{E}^i(S^n)_{u,\delta}, \mathcal{E}^j(S^{n-1})_{v,\varepsilon})$ and operators in Theorem 2.9 in cases (1)-(4)'

j	i	u	v	δ	ε	Operators	a
$i-2$	$2 \leq i \leq n-1$	$n-2i$	$n-2i+3$	1	1	$* \circ (4) \circ *$	1
	$i=n$	$u \in -n-1-2\mathbb{N}$	$3-n$	0	0	$(1)'$	$1-u-n$
		$u \in -n-2\mathbb{N}$		1	1	$* \circ (4)' \circ *$	
$i-1$	$1 \leq i \leq n$	$v-u \in 2\mathbb{N}+2$		0	0	(2)	$v-u-1$
		$v-u \in 2\mathbb{N}+1$		1	1	$* \circ (3) \circ *$	
i	$0 \leq i \leq n-1$	$v-u \in 2\mathbb{N}$		0	0	(3)	$v-u$
		$v-u \in 2\mathbb{N}+1$		1	1	$* \circ (2) \circ *$	
$i+1$	$1 \leq i \leq n-2$	0	0	0	0	(4)	1
	$i=0$	$u \in -2\mathbb{N}$		0	0	$(4)'$	$1-u$
		$u \in -1-2\mathbb{N}$		1	1	$* \circ (1)' \circ *$	
$n-i-2$	$1 \leq i \leq n-2$	0	$2i-n+3$	0	1	$* \circ (4)$	$1-u$
	$i=0$	$u \in -2\mathbb{N}$	$3-n$	0	1	$* \circ (4)'$	
		$u \in -1-2\mathbb{N}$		1	0	$(1)' \circ *$	
$n-i-1$	$0 \leq i \leq n-1$	$v-u \in (2i-n+1)+2\mathbb{N}$		0	1	$* \circ (3)$	$v-u+n-2i-1$
		$v-u \in (2i-n+2)+2\mathbb{N}$		1	0	$(2) \circ *$	
$n-i$	$1 \leq i \leq n$	$v-u \in (2i-n+1)+2\mathbb{N}$		0	1	$* \circ (2)$	$v-u+n-2i$
		$v-u \in (2i-n)+2\mathbb{N}$		1	0	$(3) \circ *$	
$n-i+1$	$2 \leq i \leq n-1$	$n-2i$	0	1	0	$(4) \circ *$	1
	$i=n$	$u \in -n-1-2\mathbb{N}$		0	1	$* \circ (1)'$	$1-u-n$
		$u \in -n-2\mathbb{N}$		1	0	$(4)' \circ *$	

where $\xi^- = \iota(-1, 0, \dots, 0)$. As is well-known, p and ι are conformal maps with the following conformal factors (see [18, Lem. 3.3] for example):

$$\begin{aligned} \iota^* g_{S^n, \iota(x)} &= \left(\frac{2}{1 + Q_n(x)} \right)^2 g_{\mathbb{R}^n, x} \quad \text{for } x \in \mathbb{R}^n, \\ p^* g_{\mathbb{R}^n, p(\omega)} &= \left(\frac{1}{1 + \omega_0} \right)^2 g_{S^n, \omega} \quad \text{for } \omega \in S^n \setminus \{[\xi^-]\}. \end{aligned}$$

In turn, the twisted pull-back of differential forms defined in (8.2) amounts to

$$\begin{aligned} (p_{u, \delta}^{(i)})^* : \mathcal{E}^i(\mathbb{R}^n) &\longrightarrow \mathcal{E}^i(S^n \setminus \{[\xi^-]\}), \quad \alpha \mapsto (1 + \omega_0)^{-u} p^* \alpha, \\ (\iota_{u, \delta}^{(i)})^* : \mathcal{E}^i(S^n) &\longrightarrow \mathcal{E}^i(\mathbb{R}^n), \quad \beta \mapsto \left(\frac{1 + Q_n(x)}{2} \right)^{-u} \iota^* \beta, \end{aligned}$$

for $u \in \mathbb{C}$, $\delta \in \mathbb{Z}/2\mathbb{Z}$, and $0 \leq i \leq n$.

Then the following proposition gives a change of coordinates in differential symmetry breaking operators.

Proposition 11.3. *Suppose a 6-tuple $(i, j, u, v, \delta, \varepsilon)$ belongs to Cases (I)-(IV') or Cases (*I)-(*IV') in Theorem 1.1, and $D = \tilde{\mathcal{D}}_{u, v+j-u-i}^{i \rightarrow j}$ (or $*_{\mathbb{R}^{n-1}} \circ \tilde{\mathcal{D}}_{u, v+j-u-i}^{i \rightarrow n-1-j}$, respectively) is a differential operator $\mathcal{E}^i(\mathbb{R}^n) \longrightarrow \mathcal{E}^j(\mathbb{R}^{n-1})$ defined as in (1.9)-(1.12). Then the compositions*

$$\begin{aligned} (p_{v, \varepsilon}^{(j)})^* \circ D \circ (\iota_{u, \delta}^{(i)})^* &: \mathcal{E}^i(S^n) \longrightarrow \mathcal{E}^j(S^{n-1}), \\ *_{S^{n-1}} \circ (p_{v-n+2j+1, \varepsilon+1}^{(n-j-1)})^* \circ D \circ (\iota_{u, \delta}^{(i)})^* &: \mathcal{E}^i(S^n) \longrightarrow \mathcal{E}^j(S^{n-1}), \end{aligned}$$

respectively, are differential symmetry breaking operators from $(\varpi_{u, \delta}^{(i)}, \mathcal{E}^i(S^n))$ to $(\varpi_{v, \varepsilon}^{(j)}, \mathcal{E}^j(S^{n-1}))$ in the coordinates of (S^n, S^{n-1}) .

In Proposition 11.3, $\iota: \mathbb{R}^n \longrightarrow S^n$ denotes the conformal compactification in the n -dimensional setting as before, but $p: S^{n-1} \setminus \{[\xi^-]\} \longrightarrow \mathbb{R}^{n-1}$ is the stereographic projection in the $(n-1)$ -dimensional setting.

The proof of Proposition 11.3 in Cases (I)-(IV') is clear. For Cases (*I)-(*IV'), we use Lemma 8.2:

$$(11.6) \quad *_{S^{n-1}} \circ (p_{v-n+2j+1, \varepsilon+1}^{(n-j-1)})^* = (p_{v, \varepsilon}^{(j)})^* \circ *_{\mathbb{R}^{n-1}} \quad \text{on } \mathcal{E}^{n-j-1}(\mathbb{R}^{n-1}).$$

We end this section by giving some few examples of $(p_{v, \varepsilon}^{(j)})^* \circ D \circ (\iota_{u, \delta}^{(i)})^*$ from Lemmas 8.5, 8.9, and 8.11. The last one is related to the factorization identity, which we see in Theorem 13.18 (4).

j	u	δ	v	ε	D	$\left(p_{v,\varepsilon}^{(j)}\right)^* \circ D \circ \left(\iota_{u,\delta}^{(i)}\right)^*$
$i-1$	u	1	$u+1$	1	$\tilde{\mathcal{D}}_{u,0}^{i \rightarrow i-1} = \text{Rest}_{x_n=0} \circ \iota_{\frac{\partial}{\partial x_n}}$	$\text{Rest}_{S^{n-1}} \circ \iota_{N_{S^{n-1}}(S^n)}$
i	u	0	u	0	$\tilde{\mathcal{D}}_{u,0}^{i \rightarrow i} = \text{Rest}_{x_n=0}$	$\text{Rest}_{S^{n-1}}$
$i+1$	0	0	0	0	$\tilde{\mathcal{D}}_{0,0}^{i \rightarrow i+1} = \text{Rest}_{x_n=0} \circ d_{\mathbb{R}^n}$	$\text{Rest}_{S^{n-1}} \circ d_{S^n}$
$i-1$	$n-2i$	0	$n-2i+2$	0	$\tilde{\mathcal{D}}_{n-2i,1}^{i \rightarrow i-1} = -\text{Rest}_{x_n=0} \circ d_{\mathbb{R}^n}^*$	$-\text{Rest}_{S^{n-1}} \circ d_{S^n}^*$
$i-2$	$n-2i$	1	$n-2i+3$	1	$\tilde{\mathcal{D}}_{n-2i,1}^{i \rightarrow i-2} = \text{Rest}_{x_n=0} \circ \iota_{\frac{\partial}{\partial x_n}} \circ d_{\mathbb{R}^n}^*$	$\text{Rest}_{S^{n-1}} \circ \iota_{N_{S^{n-1}}(S^n)} \circ d_{S^n}^*$

12. INTERTWINING OPERATORS

In this chapter we determine all conformally covariant differential operators between the spaces of differential forms on the standard Riemannian sphere S^n , and thus solve Problems A and B in the case where $X = Y = S^n$. We note that the case $X = Y$ (and $G = G'$) is much easier than the case $X \supsetneq Y$ which we have discussed in Chapters 6-11.

We have seen in Proposition 8.6 that the differential $d: \mathcal{E}^i(X) \rightarrow \mathcal{E}^{i+1}(X)$ (the codifferential $d^*: \mathcal{E}^{i+1}(X) \rightarrow \mathcal{E}^i(X)$, respectively) intertwines two representations $\varpi_{u,\delta}^{(i)}$ and $\varpi_{v,\varepsilon}^{(i+1)}$ (see (1.1)) of the conformal group of any oriented pseudo-Riemannian manifold X for appropriate twisting parameters (u, δ) and (v, ε) , respectively. Conversely, our classification (Theorem 12.1) shows that d is the unique differential operator from $\mathcal{E}^i(S^n)$ to $\mathcal{E}^{i+1}(S^n)$ (up to scalar multiplication) that commutes with conformal diffeomorphisms of S^n . Similarly, we shall prove that the codifferential d^* is characterized as the unique differential operator (up to scalar multiplication) $\mathcal{E}^{i+1}(S^n) \rightarrow \mathcal{E}^i(S^n)$ that intertwines twisted representations of the conformal group of S^n . On the other hand, we find countably many bases of conformally covariant differential operators of higher order that map $\mathcal{E}^i(S^n)$ into $\mathcal{E}^j(S^n)$ when $j = i$ (see Theorem 12.1).

One could give a proof of those results by combining the algebraic results on the classification of homomorphisms between generalized Verma modules by Boe–Collingwood [2] with the geometric construction of differential operators by Branson [4], although the existing literature treats only connected groups and one needs some extra work to discuss disconnected groups. Alternatively, we shall give a self-contained proof of these results from scratch by the matrix-valued F-method. We know we could shorten a significant part of the proof (*e.g.* the relationship between λ and μ) if we used some elementary results on Verma modules. Instead we provide an alternative approach, as this baby example might be illustrative about the use of the F-method in a more general matrix-valued setting.

12.1. Classification of differential intertwining operators between forms on S^n . Let $0 \leq i \leq n$. For $\ell \in \mathbb{N}_+$, define a differential operator (*Branson's operator*)

$$\mathcal{T}_{2\ell}^{(i)}: \mathcal{E}^i(\mathbb{R}^n) \rightarrow \mathcal{E}^i(\mathbb{R}^n)$$

by

$$\begin{aligned} (12.1) \quad \mathcal{T}_{2\ell}^{(i)} &:= \left(\left(\frac{n}{2} - i - \ell \right) d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* + \left(\frac{n}{2} - i + \ell \right) d_{\mathbb{R}^n}^* d_{\mathbb{R}^n} \right) \Delta_{\mathbb{R}^n}^{\ell-1} \\ &= \left(-2\ell d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* - \left(\frac{1}{2}n + \ell - i \right) \Delta_{\mathbb{R}^n} \right) \Delta_{\mathbb{R}^n}^{\ell-1}. \end{aligned}$$

Then the following theorem is the main result of this chapter.

Theorem 12.1. *Let $n \geq 2$.*

(1) *Let $0 \leq i \leq n$ and $\ell \in \mathbb{N}_+$. We set*

$$u := \frac{n}{2} - i - \ell, \quad v := \frac{n}{2} - i + \ell.$$

Then the differential operator $\mathcal{T}_{2\ell}^{(i)}$ extends to the conformal compactification S^n of \mathbb{R}^n , and induces a nonzero $O(n+1, 1)$ -homomorphism $\mathcal{E}^i(S^n)_{u,\delta} \longrightarrow \mathcal{E}^i(S^n)_{v,\delta}$ for $\delta \in \mathbb{Z}/2\mathbb{Z}$, to be denoted simply by the same letter $\mathcal{T}_{2\ell}^{(i)}$.

(2) *Let $0 \leq i, j \leq n$, $(u, v) \in \mathbb{C}^2$ and $(\delta, \varepsilon) \in (\mathbb{Z}/2\mathbb{Z})^2$. Then the space of conformally covariant differential operators, $\text{Diff}_{O(n+1,1)}(\mathcal{E}^i(S^n)_{u,\delta}, \mathcal{E}^j(S^n)_{v,\varepsilon})$, is at most one-dimensional. More precisely, this space is nonzero in the eight cases listed below. The corresponding generators are given as follows:*

Case a. $0 \leq i \leq n$, $u \in \mathbb{C}$, $\delta \in \mathbb{Z}/2\mathbb{Z}$.

$$\text{id}: \mathcal{E}^i(S^n)_{u,\delta} \longrightarrow \mathcal{E}^i(S^n)_{u,\delta}.$$

Case b. $0 \leq i \leq n-1$, $\delta \in \mathbb{Z}/2\mathbb{Z}$.

$$d: \mathcal{E}^i(S^n)_{0,\delta} \longrightarrow \mathcal{E}^{i+1}(S^n)_{0,\delta}.$$

Case c. $1 \leq i \leq n$, $\delta \in \mathbb{Z}/2\mathbb{Z}$.

$$d^*: \mathcal{E}^i(S^n)_{n-2i,\delta} \longrightarrow \mathcal{E}^{i-1}(S^n)_{n-2i+2,\delta}.$$

Case d. $0 \leq i \leq n$, $\ell \in \mathbb{N}_+$, $\delta \in \mathbb{Z}/2\mathbb{Z}$.

$$\mathcal{T}_{2\ell}^{(i)}: \mathcal{E}^i(S^n)_{\frac{n}{2}-\ell-i,\delta} \longrightarrow \mathcal{E}^i(S^n)_{\frac{n}{2}+\ell-i,\delta}.$$

Case *a. $0 \leq i \leq n$, $u \in \mathbb{C}$ and $\delta \in \mathbb{Z}/2\mathbb{Z}$.

$$*: \mathcal{E}^i(S^n)_{u,\delta} \longrightarrow \mathcal{E}^{n-i}(S^n)_{u-n+2i,\delta+1}.$$

Case *b. $0 \leq i \leq n-1$, $\delta \in \mathbb{Z}/2\mathbb{Z}$.

$$* \circ d: \mathcal{E}^i(S^n)_{0,\delta} \longrightarrow \mathcal{E}^{n-i-1}(S^n)_{2i+2-n,\delta+1}.$$

Case *c. $1 \leq i \leq n$, $\delta \in \mathbb{Z}/2\mathbb{Z}$.

$$d \circ *: \mathcal{E}^i(S^n)_{n-2i,\delta} \longrightarrow \mathcal{E}^{n-i+1}(S^n)_{0,\delta+1}.$$

Case *d. $0 \leq i \leq n$, $\ell \in \mathbb{N}_+$, $\delta \in \mathbb{Z}/2\mathbb{Z}$.

$$* \circ \mathcal{T}_{2\ell}^{(i)}: \mathcal{E}^i(S^n)_{\frac{n}{2}-\ell-i,\delta} \longrightarrow \mathcal{E}^{n-i}(S^n)_{-\frac{n}{2}+\ell+i,\delta+1}.$$

We shall give a proof of Theorem 12.1 in Section 12.7.

12.2. Differential symmetry breaking operators between principal series representations. We reformulate the problem in terms of representation theory. Let $I(i, \lambda)_\alpha$ be the principal series representation of the Lorentz group $G = O(n+1, 1)$. We determine differential symmetry breaking operators between $I(i, \lambda)_\alpha$ s as follows:

Theorem 12.2. *Let $n \geq 2$, $0 \leq i, j \leq n$, $(\lambda, \nu) \in \mathbb{C}^2$ and $(\alpha, \beta) \in (\mathbb{Z}/2\mathbb{Z})^2$.*

- (1) *The following three conditions on the 6-tuple $(i, j, \lambda, \nu, \alpha, \beta)$ are equivalent:*
 - (i) $\text{Diff}_{O(n+1,1)}(I(i, \lambda)_\alpha, I(j, \nu)_\beta) \neq \{0\}$.
 - (ii) $\dim_{\mathbb{C}} \text{Diff}_{O(n+1,1)}(I(i, \lambda)_\alpha, I(j, \nu)_\beta) = 1$.
 - (iii) *The 6-tuple belongs to one of the following:*
 - Case 1. $j = i + 1$, $(\lambda, \nu) = (i, i + 1)$, and $\alpha \equiv \beta + 1 \pmod{2}$;
 - Case 2. $j = i - 1$, $(\lambda, \nu) = (n - i, n - i + 1)$, and $\alpha \equiv \beta + 1 \pmod{2}$;
 - Case 3. $j = i$, $\lambda + \nu = n$, $\nu - \lambda \in 2\mathbb{N}_+$, and $\alpha \equiv \beta \pmod{2}$;
 - Case 4. $j = i$, $\lambda = \nu$, and $\alpha \equiv \beta \pmod{2}$.
- (2) *Any differential G -intertwining operators from $I(i, \lambda)_\alpha$ to $I(j, \nu)_\beta$ are proportional to the following differential operators $\mathcal{E}^i(\mathbb{R}^n) \longrightarrow \mathcal{E}^j(\mathbb{R}^n)$ in the flat picture:*
 - Case 1. d ;
 - Case 2. d^* ;
 - Case 3. $\mathcal{T}_{\nu-\lambda}^{(i)} = \left(\frac{1}{2}(n - 2i - \nu + \lambda)d_{\mathbb{R}^n}d_{\mathbb{R}^n}^* + \frac{1}{2}(n - 2i + \nu - \lambda)d_{\mathbb{R}^n}^*d_{\mathbb{R}^n}\right)\Delta_{\mathbb{R}^n}^{\frac{1}{2}(\nu-\lambda)-1}$;
 - Case 4. id .

For the proof we apply the F-method in the special case where $G = G' = O(n+1, 1)$. For $0 \leq i \leq n$, $\alpha \in \mathbb{Z}/2\mathbb{Z}$, and $\lambda \in \mathbb{C}$, we denote by $\sigma_{\lambda, \alpha}^{(i)}$ the outer tensor product representation $\bigwedge^i(\mathbb{C}^n) \boxtimes (-1)^\alpha \boxtimes \mathbb{C}_\lambda$ of the Levi subgroup $L = MA \simeq O(n) \times O(1) \times \mathbb{R}$ on the i -th exterior tensor space $\bigwedge^i(\mathbb{C}^n)$. We recall that the principal series representation $I(i, \lambda)_\alpha$ of $G = O(n+1, 1)$ is the unnormalized induction from the representation $\sigma_{\lambda, \alpha}^{(i)}$ of P with trivial action by N_+ .

By Fact 3.3 we have a bijection:

$$(12.2) \quad \text{Diff}_{O(n+1,1)}(I(i, \lambda)_\alpha, I(j, \nu)_\beta) \xrightarrow{\sim} \text{Sol}\left(\mathfrak{n}_+; \sigma_{\lambda, \alpha}^{(i)}, \sigma_{\nu, \beta}^{(j)}\right),$$

where the right-hand side is given by Lemma 3.4 as

$$\left\{ \psi \in \text{Hom}_L\left(\sigma_{\lambda, \alpha}^{(i)}, \sigma_{\nu, \beta}^{(j)} \otimes \text{Pol}[\zeta_1, \dots, \zeta_n]\right) : \widehat{d\pi_{(i, \lambda)^*}}(N_1^+)\psi = 0 \right\}.$$

We recall from (5.7)–(5.9) and (5.11) that $H_{i \rightarrow j}^{(k)}$ and $\widetilde{H}_{i \rightarrow i}^{(2)}$ are $\text{Hom}_G(\bigwedge^i(\mathbb{C}^n), \bigwedge^j(\mathbb{C}^n))$ -valued harmonic polynomials. Then the following proposition holds:

Proposition 12.3. *Suppose $n \geq 2$. Let the 6-tuple $(i, j, \lambda, \nu, \alpha, \beta)$ be as in Cases 1-4 of Theorem 12.2. Then,*

$$\begin{aligned} & \text{Sol} \left(\mathbf{n}_+; \sigma_{\lambda, \alpha}^{(i)}, \sigma_{\nu, \beta}^{(j)} \right) \\ &= \begin{cases} \mathbb{C} H_{i \rightarrow i+1}^{(1)} & \text{Case 1,} \\ \mathbb{C} H_{i \rightarrow i-1}^{(1)} & \text{Case 2,} \\ \mathbb{C} \left(-\frac{1}{2}(n + \nu - \lambda) \left(1 - \frac{2i}{n} \right) Q_n^{\frac{\nu-\lambda}{2}} H_{i \rightarrow i}^{(0)} + (\nu - \lambda) Q_n^{\frac{\nu-\lambda}{2}-1} \tilde{H}_{i \rightarrow i}^{(2)} \right) & \text{Case 3,} \\ \mathbb{C} H_{i \rightarrow i}^{(0)} & \text{Case 4,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

We note that $\tilde{H}_{i \rightarrow i}^{(2)} = 0$ for $i = 0, n$.

We shall give a proof of Proposition 12.3 in Sections 12.3 to 12.6. Admitting Proposition 12.3, we first complete the proof of Theorem 12.2.

Proof of Theorem 12.2. The first statement is a direct consequence of Proposition 12.3 and the bijection (12.2). To see the second statement, we recall from Fact 3.3 that the bijection (12.2) is given by the symbol map if we use the flat coordinates. Since $\text{Symb}(d_{\mathbb{R}^n}) = H_{i \rightarrow i+1}^{(1)}$, $\text{Symb}(d_{\mathbb{R}^n}^*) = H_{i \rightarrow i-1}^{(1)}$, and $\text{Symb}(\text{id}) = H_{i \rightarrow i}^{(0)}$ by Lemma 8.23, the second statement in Cases 1, 2, and 4 is verified.

In Case 3, we need a supplementary computation. Indeed, we apply Lemma 8.23 (3) and (4) to get the formula

$$\text{Symb} \left(\left(-A + \left(\frac{i}{n} - 1 \right) B \right) d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* + \left(-A + \frac{i}{n} B \right) d_{\mathbb{R}^n}^* d_{\mathbb{R}^n} \right) = A Q_n H_{i \rightarrow i}^{(0)} + B \tilde{H}_{i \rightarrow i}^{(2)}.$$

By putting $A = -\frac{1}{2}(n + \nu - \lambda) \left(1 - \frac{2i}{n} \right)$ and $B = \nu - \lambda$, we have

$$\text{Symb}(u d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* + v d_{\mathbb{R}^n}^* d_{\mathbb{R}^n}) = -\frac{1}{2}(n + \nu - \lambda) \left(1 - \frac{2i}{n} \right) Q_n H_{i \rightarrow i}^{(0)} + (\nu - \lambda) \tilde{H}_{i \rightarrow i}^{(2)},$$

where $u = \frac{1}{2}(n - 2i - \nu + \lambda)$ and $v = \frac{1}{2}(n - 2i + \nu - \lambda)$. Thus the second statement in Case 3 is also verified. \square

12.3. Description of $\text{Hom}_L(V, W \otimes \text{Pol}(\mathbf{n}_+))$. In order to prove Proposition 12.3, we begin with an elementary algebraic lemma.

Lemma 12.4.

$$\begin{aligned} & \text{Hom}_L \left(\sigma_{\lambda, \alpha}^{(i)}, \sigma_{\nu, \beta}^{(j)} \otimes \text{Pol}[\zeta_1, \dots, \zeta_n] \right) \\ &= \begin{cases} \mathbb{C} Q_n^{\frac{\nu-\lambda-1}{2}} H_{i \rightarrow i+1}^{(1)} & \text{if } j = i+1, \nu - \lambda \in 2\mathbb{N} + 1, & \beta \equiv \alpha + 1 \pmod{2}, \\ \mathbb{C} Q_n^{\frac{\nu-\lambda-1}{2}} H_{i \rightarrow i-1}^{(1)} & \text{if } j = i-1, \nu - \lambda \in 2\mathbb{N} + 1, & \beta \equiv \alpha + 1 \pmod{2}, \\ \mathbb{C} Q_n^{\frac{\nu-\lambda}{2}} H_{i \rightarrow i}^{(0)} + \mathbb{C} Q_n^{\frac{\nu-\lambda}{2}-1} \tilde{H}_{i \rightarrow i}^{(2)} & \text{if } j = i \in \{1, \dots, n-1\}, \nu - \lambda \in 2\mathbb{N}_+, & \beta \equiv \alpha \pmod{2}, \\ \mathbb{C} Q_n^{\frac{\nu-\lambda}{2}} H_{0 \rightarrow 0}^{(0)} & \text{if } j = i \in \{0, n\}, \nu - \lambda \in 2\mathbb{N}_+, & \beta \equiv \alpha \pmod{2}, \\ \mathbb{C} H_{i \rightarrow i}^{(0)} & \text{if } j = i, \nu = \lambda, & \beta \equiv \alpha \pmod{2}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Proof. We may restrict ourselves to homogeneous polynomials because L preserves the degree of homogeneity in $\text{Pol}(\mathfrak{n}_+)$. We consider the action of the second and third factors of $L \simeq O(n) \times O(1) \times \mathbb{R}$. Since $e^{tH_0} \in A$ and $-1 \in O(1)$ act on $\mathfrak{n}_+ \simeq \mathbb{C}^n$ as the scalars e^t and -1 , respectively, we conclude

$$\text{Hom}_{O(1) \times A} \left(\sigma_{\lambda, \alpha}^{(i)}, \sigma_{\nu, \beta}^{(j)} \otimes \text{Pol}^a(\mathfrak{n}_+) \right) \neq \{0\}$$

if and only if

$$\nu = \lambda + a \quad \text{and} \quad \beta \equiv \alpha + a \pmod{2}.$$

In this case, we have

$$\begin{aligned} \text{Hom}_L \left(\sigma_{\lambda, \alpha}^{(i)}, \sigma_{\nu, \beta}^{(j)} \otimes \text{Pol}^a[\zeta_1, \dots, \zeta_n] \right) &\simeq \text{Hom}_{O(n)} \left(\bigwedge^i(\mathbb{C}^n), \bigwedge^j(\mathbb{C}^n) \otimes \text{Pol}^a[\zeta_1, \dots, \zeta_n] \right) \\ &\simeq \bigoplus_{\substack{0 \leq k \leq a \\ k \equiv a \pmod{2}}} \text{Hom}_{O(n)} \left(\bigwedge^i(\mathbb{C}^n), \bigwedge^j(\mathbb{C}^n) \otimes \mathcal{H}^k(\mathbb{C}^n) \right) \end{aligned}$$

because we have an $O(n)$ -isomorphism:

$$\text{Pol}[\zeta_1, \dots, \zeta_n] \simeq \text{Pol}[Q_n(\zeta)] \otimes \left(\bigoplus_{k=0}^{\infty} \mathcal{H}^k(\mathbb{C}^n) \right).$$

Now Lemma follows from Lemma 5.6 and Proposition 5.7. \square

In order to prove Proposition 12.3, it is sufficient to find ψ ($\neq 0$) that belongs to the right-hand side of the identity in Lemma 12.4 satisfying $\widehat{d\pi_{(i,\lambda)^*}}(N_1^+) \psi = 0$. We shall carry out this computation in the next sections.

12.4. Solving the F-system when $j = i+1$. This section treats the case $j = i+1$. We shall use I, I' to denote elements in $\mathcal{I}_{n,i}$ and \tilde{I} for those in $\mathcal{I}_{n,i+1}$. This is slightly different from the convention for index sets adopted in the previous chapters.

According to Lemma 12.4, we may assume $\nu - \lambda = 2\ell + 1$ for some $\ell \in \mathbb{N}$ and $\beta \equiv \alpha + 1 \pmod{2}$. We set $\psi = Q_n^\ell H_{i \rightarrow i+1}^{(1)}$. With respect to the standard basis $\{e_I : I \in \mathcal{I}_{n,i}\}$ of $\Lambda^i(\mathbb{C}^n)$ and $\{e_{\tilde{I}} : \tilde{I} \in \mathcal{I}_{n,i+1}\}$ of $\Lambda^{i+1}(\mathbb{C}^n)$, we set, as in Section 4.5,

$$\begin{aligned}\psi_{I\tilde{I}}(\zeta) &:= Q_n(\zeta)^\ell \left(H_{i \rightarrow i+1}^{(1)} \right)_{I\tilde{I}}(\zeta), \\ M_{I\tilde{I}} &:= \left\langle \widehat{d\pi_{(i,\lambda)}^*}(N_1^+) \psi(e_I), e_{\tilde{I}}^\vee \right\rangle.\end{aligned}$$

Then the proof of Proposition 12.3 for $j = i + 1$ reduces to the following lemma:

Lemma 12.5. *Suppose $0 \leq i \leq n - 1$. Then the following three conditions are equivalent:*

- (i) $\widehat{d\pi_{(i,\lambda)}^*}(N_1^+) \psi = 0$.
- (ii) $M_{I\tilde{I}} = 0$ for all $I \in \mathcal{I}_{n,i}$ and $\tilde{I} \in \mathcal{I}_{n,i+1}$.
- (iii) $(\lambda, \nu) = (i, i + 1)$ and $\ell = 0$.

Let us verify this lemma. According to the decomposition of $\widehat{d\pi_{(i,\lambda)}^*}(N_1^+)$ into the scalar and vector parts, we decompose $M_{I\tilde{I}}$ its matrix components $M_{I\tilde{I}} = M_{I\tilde{I}}^{\text{scalar}} + M_{I\tilde{I}}^{\text{vect}}$ as in Proposition 4.9, where

$$\begin{aligned}M_{I\tilde{I}}^{\text{scalar}} &= \widehat{d\pi_{\lambda}^*}(N_1^+) \psi_{I\tilde{I}} = \left(\lambda \frac{\partial}{\partial \zeta_1} + E_\zeta \frac{\partial}{\partial \zeta_1} - \frac{1}{2} \zeta_1 \Delta_{\mathbb{C}^n} \right) \psi_{I\tilde{I}}, \\ M_{I\tilde{I}}^{\text{vect}} &= \sum_{I' \in \mathcal{I}_{n,i}} A_{II'} \psi_{I'\tilde{I}}.\end{aligned}$$

Lemma 12.6. *For $I \in \mathcal{I}_{n,i}$ and $\tilde{I} \in \mathcal{I}_{n,i+1}$, we have*

$$\psi_{I\tilde{I}} = \begin{cases} Q_n^\ell(\zeta) \operatorname{sgn}(I; p) \zeta_p & \text{if } \tilde{I} = I \cup \{p\}, \\ 0 & \text{if } I \not\subset \tilde{I}. \end{cases}$$

Proof. Clear from the definition of $H_{i \rightarrow i+1}^{(1)}$ given in (5.9). □

Lemma 12.7. *For $I \in \mathcal{I}_{n,i}$ and $\tilde{I} \in \mathcal{I}_{n,i+1}$, we have*

$$M_{I\tilde{I}}^{\text{scalar}} = \begin{cases} 0 & \text{if } I \not\subset \tilde{I}, \\ \ell(2\lambda + 2\ell - n) \zeta_1^2 Q_n^{\ell-1}(\zeta) + (\lambda + 2\ell) Q_n^\ell(\zeta) & \text{if } \tilde{I} \setminus I = \{1\}, \\ \ell(2\lambda + 2\ell - n) \operatorname{sgn}(I; p) \zeta_1 \zeta_p Q_n^{\ell-1}(\zeta) & \text{if } \tilde{I} \setminus I = \{p\}, p \neq 1. \end{cases}$$

Proof. Since $\Delta_{\mathbb{C}^n} Q_n^\ell(\zeta) = 2\ell(2\ell - 2 + n) Q_n^{\ell-1}(\zeta)$, we have the identity:

$$(12.3) \quad \widehat{d\pi_{\lambda}^*}(N_1^+) Q_n^\ell(\zeta) = \ell(2\lambda + 2\ell - n) \zeta_1 Q_n^{\ell-1}(\zeta).$$

Then the lemma follows from Lemma 4.6. □

By the formula of $A_{II'}$ (see Lemma 5.3), we have

$$(12.4) \quad M_{I\tilde{I}}^{\text{vect}} = \begin{cases} \sum_{q \in I} \text{sgn}(I; q) \frac{\partial}{\partial \zeta_q} \psi_{I \setminus \{q\} \cup \{1\}, \tilde{I}} & \text{if } 1 \notin I, \\ \sum_{q \notin I} \text{sgn}(I; q) \frac{\partial}{\partial \zeta_q} \psi_{I \setminus \{1\} \cup \{q\}, \tilde{I}} & \text{if } 1 \in I. \end{cases}$$

Combining Lemma 12.7 with an easy computation of $M_{I\tilde{I}}^{\text{vect}}$, we get the following.

Lemma 12.8. *Let $n \geq 2$, $0 \leq i \leq n-1$, $I \in \mathcal{I}_{n,i}$ and $\tilde{I} \in \mathcal{I}_{n,i+1}$.*

(1) *Assume $1 \in I$ and $\tilde{I} = I \cup \{p\}$ for some $p \notin I$. Then*

$$M_{I\tilde{I}} = \ell(2\lambda + 2\ell - n + 2) \text{sgn}(I; p) \zeta_1 \zeta_p Q_n^{\ell-1}(\zeta).$$

(2) *Assume $1 \notin I$ and $\tilde{I} = I \cup \{1\}$. Then*

$$M_{I\tilde{I}} = (\lambda - i + 2\ell) Q_n^\ell(\zeta) + (2\lambda + 2\ell - n) \ell \zeta_1^2 Q_n^{\ell-1}(\zeta) - 2\ell Q_I(\zeta) Q_n^{\ell-1}(\zeta).$$

(3) *Assume $\tilde{I} = K \cup \{1, q\}$, $I = K \cup \{p\}$ with $1 \neq p \neq q \neq 1$. Then,*

$$M_{I\tilde{I}} = -2\ell \text{sgn}(K; p, q) \zeta_p \zeta_q Q_n^{\ell-1}(\zeta).$$

(4) *Assume $1 \notin \tilde{I}$ and $\tilde{I} = I \cup \{p\}$. Then*

$$M_{I\tilde{I}} = \ell(2\lambda + 2\ell - 2) \text{sgn}(I; p) \zeta_1 \zeta_p Q_n^{\ell-1}(\zeta).$$

(5) *Otherwise, $M_{I\tilde{I}} = 0$.*

We are ready to give a proof of Lemma 12.5, and consequently Proposition 12.3 for $j = i + 1$.

Proof of Lemma 12.5. Suppose $M_{I\tilde{I}} = 0$ for all $I \in \mathcal{I}_{n,i}$ and $\tilde{I} \in \mathcal{I}_{n,i+1}$. Then ℓ must be zero, as is seen from Lemma 12.8 (3) which works for $i \neq 0$ or from Lemma 12.8 (4) which works for $i = 0$ and $n \geq 2$. In turn, Lemma 12.8 (2) implies $\lambda = i$, and thus $\nu = i + 1$.

Conversely, if $\ell = 0$ and $(\lambda, \nu) = (i, i + 1)$, then clearly $M_{I\tilde{I}} = 0$ for all I and \tilde{I} by Lemma 12.8. Thus Lemma 12.5 is proved. \square

12.5. Solving the F-system when $j = i$. In this section we treat the case $j = i$ and $\beta - \alpha \equiv 0 \pmod{2}$. According to Lemma 12.4, we need to consider the case $\nu - \lambda \in 2\mathbb{N}$. Since the case $\nu = \lambda$ is easy, let us assume $\nu - \lambda \in 2\mathbb{N}_+$. We set

$$\ell := \frac{1}{2}(\nu - \lambda) - 1 \in \mathbb{N}.$$

First consider that $j = i = 0$. Then any element in $\text{Hom}_L \left(\sigma_{\lambda, \alpha}^{(i)}, \sigma_{\nu, \beta}^{(j)} \otimes \text{Pol}[\zeta_1, \dots, \zeta_n] \right)$ is proportional to $Q_n^{\ell+1} H_{0 \rightarrow 0}^{(0)}$ by Lemma 12.4. We set $\psi := Q_n^{\ell+1} H_{0 \rightarrow 0}^{(0)}$. Then we have the following.

Lemma 12.9. *Suppose $i = 0$ or n and $\nu - \lambda = 2\ell + 2$ with $\ell \in \mathbb{N}$. Then the following two conditions are equivalent:*

- (i) $\widehat{d\pi_{(i,\lambda)*}}(N_1^+)\psi = 0$.
- (ii) $(\lambda, \nu) = (\frac{n}{2} - \ell - 1, \frac{n}{2} + \ell + 1)$.

Proof. Since $i = 0$ or n , there is no “vector part” of $\widehat{d\pi_{(i,\lambda)*}}(N_1^+) = \widehat{d\pi_{\lambda*}}(N_1^+)$, and thus

$$\widehat{d\pi_{(i,\lambda)*}}(N_1^+)\psi(\zeta) = (\ell + 1)(2\lambda + 2\ell - n + 2)\zeta_1 Q_n^\ell(\zeta) H_{0 \rightarrow 0}^{(0)}$$

by (12.3). Now the lemma is clear. \square

From now, we assume $i \neq 0$ and $n \geq 2$. For $A, B \in \mathbb{C}$ we set

$$(12.5) \quad \begin{aligned} \psi &:= A Q_n^{\ell+1} H_{i \rightarrow i}^{(0)} + B Q_n^\ell \tilde{H}_{i \rightarrow i}^{(2)} \in \text{Hom}_L \left(\sigma_{\lambda, \alpha}^{(i)}, \sigma_{\nu, \beta}^{(j)} \otimes \text{Pol}[\zeta_1, \dots, \zeta_n] \right), \\ M_{II'} &:= \langle \widehat{d\pi_{(i,\lambda)*}}(N_1^+)\psi(e_I), e_{I'}^\vee \rangle \quad \text{for } I, I' \in \mathcal{I}_{n,i}. \end{aligned}$$

Lemma 12.10. *Let $1 \leq i \leq n - 1$, and $\nu - \lambda = 2\ell + 2$ with $\ell \in \mathbb{N}$. Suppose $(A, B) \neq (0, 0)$. Then the following three conditions are equivalent:*

- (i) $\widehat{d\pi_{(i,\lambda)*}}(N_1^+)\psi = 0$.
- (ii) $M_{II'} = 0$ for all $I, I' \in \mathcal{I}_{n,i}$.
- (iii) $(\lambda, \nu) = (\frac{n}{2} - \ell - 1, \frac{n}{2} + \ell + 1)$ and (A, B) is proportional to $(-\frac{1}{2}(n + \nu - \lambda)(1 - \frac{2i}{n}), \nu - \lambda)$.

The equivalence (i) \Leftrightarrow (ii) is obvious, and we shall prove the equivalence (ii) \Leftrightarrow (iii) after Lemma 12.13 where we compute $M_{II'}$ explicitly. For this we use a couple of lemmas as follows.

Lemma 12.11. *Let ψ be given as in (12.5). Then, for $I, I' \in \mathcal{I}_{n,i}$, we have*

$$\psi_{II'}(\zeta) = \begin{cases} A Q_n^{\ell+1}(\zeta) + B \tilde{Q}_I(\zeta) Q_n^\ell(\zeta) & \text{if } I = I', \\ B \text{sgn}(K; p, q) Q_n^\ell(\zeta) \zeta_p \zeta_q & \text{if } I = K \cup \{p\} \text{ and } I' = K \cup \{q\}, \\ 0 & \text{otherwise,} \end{cases}$$

where we recall $\tilde{Q}_I(\zeta) = \sum_{\ell \in I} \zeta_\ell^2 - \frac{i}{n} Q_n(\zeta)$ from (5.12).

Proof. Clear from the definitions of $H_{i \rightarrow i}^{(0)}$ and $\tilde{H}_{i \rightarrow i}^{(2)}$ given in (5.7) and (5.11). \square

Lemma 12.12. *For $I, I' \in \mathcal{I}_{n,i}$, the scalar part $M_{II'}^{\text{scalar}}$ is given as follows.*

- (1) $I = K \cup \{p\}, I' = K \cup \{q\}$:

$$M_{II'}^{\text{scalar}} = \begin{cases} B \text{sgn}(K; p) (\ell(2\lambda + 2\ell - n) \zeta_1^2 \zeta_p Q_n^{\ell-1}(\zeta) + (\lambda + 2\ell + 1) \zeta_p Q_n^\ell(\zeta)) & \text{if } q = 1, \\ B \text{sgn}(K; p, q) (\ell(2\lambda + 2\ell - n) \zeta_1 \zeta_p \zeta_q Q_n^{\ell-1}(\zeta)) & \text{if } q \neq 1. \end{cases}$$

(2) $I = I' : \text{Suppose that } I = K \cup \{p\}. \text{ Then,}$

$$M_{II'}^{\text{scalar}} = \begin{cases} \zeta_1 Q_n^{\ell-1}(\zeta) (B\ell(a-2)Q_I(\zeta) + \{aA(\ell+1) - \frac{B}{n}((n+a)(n-i) + \ell(2n-ia))\} Q_n(\zeta)) & \text{if } p = 1, \\ \zeta_1 Q_n^{\ell-1}(\zeta) (B\ell(a-2)Q_I(\zeta) + \{aA(\ell+1) - \frac{iB}{n}(a(\ell+1) + n)\} Q_n(\zeta)) & \text{if } p \neq 1. \end{cases}$$

Proof. Direct computation by using Lemma 12.11, (12.3) and Lemma 4.6. \square

We set

$$\begin{aligned} a &:= 2\lambda + 2\ell - n + 2, \\ b &:= 2A(\ell+1) + B\left(\frac{n}{2} + \ell + 1\right)\left(1 - \frac{2i}{n}\right). \end{aligned}$$

Lemma 12.13. *Let ψ be as in (12.5) and $I, I' \in \mathcal{I}_{n,i}$.*

(1) *Assume $1 \notin I = I'$. Then*

$$M_{II'} = a\zeta_1 Q_n^{\ell-1}(\zeta) \left((\ell+1) \left(A - \frac{iB}{n} \right) Q_n(\zeta) + B\ell Q_I(\zeta) \right).$$

(2) *Assume $1 \in I = I'$. Then*

$$M_{II'} = a\zeta_1 Q_n^{\ell-1} \left(\left(A(\ell+1) + B \left(1 - \frac{i(\ell+1)}{n} \right) \right) Q_n + B\ell Q_I \right).$$

(3) *Assume $I = K \cup \{1\}, I' = K \cup \{p\}$ with $p \neq 1$. Then*

$$M_{II'} = \text{sgn}(K; p) \zeta_p Q_n^{\ell-1} \left(a\ell B \zeta_1^2 + \left(\frac{1}{2}aB - b \right) Q_n \right).$$

(4) *Assume $I = K \cup \{p\}, I' = K \cup \{1\}$ with $p \neq 1$. Then*

$$M_{II'} = \text{sgn}(K; p) \zeta_p Q_n^{\ell-1} \left(a\ell B \zeta_1^2 + \left(\frac{1}{2}aB + b \right) Q_n \right).$$

(5) *Assume $I = K \cup \{p\}, I' = K \cup \{q\}$ with $p \neq q, 1 \notin I$, and $1 \notin I'$. Then*

$$M_{II'} = aB\ell \text{sgn}(K; p, q) \zeta_1 \zeta_p \zeta_q Q_n^{\ell-1}.$$

(6) *Otherwise, $M_{II'} = 0$.*

Proof. We give a proof for (1). Suppose $1 \notin I = I'$. It follows from Lemmas 5.3 and 12.11 that

$$\begin{aligned} M_{II'}^{\text{vect}} &= \sum_{k \in I} \left(\text{sgn}(I; k) \frac{\partial}{\partial \zeta_k} \right) (B \text{sgn}(I; 1, k) Q_n^\ell(\zeta) \zeta_1 \zeta_k) \\ &= B \zeta_1 \sum_{k \in I} \frac{\partial}{\partial \zeta_k} (Q_n^\ell(\zeta) \zeta_k) \\ &= B \zeta_1 (2\ell Q_n^{\ell-1}(\zeta) Q_I(\zeta) + i Q_n^\ell(\zeta)). \end{aligned}$$

Together with the formula of $M_{II'}^{\text{scalar}}$ given in Lemma 12.12, we have

$$\begin{aligned} M_{II'} &= M_{II'}^{\text{scalar}} + M_{II'}^{\text{vector}} \\ &= a \zeta_1 Q_n^{\ell-1}(\zeta) \left((\ell + 1) \left(A - \frac{iB}{n} \right) Q_n(\zeta) + B \ell Q_I(\zeta) \right). \end{aligned}$$

Thus the first assertion is proved. The other cases are similar and omitted. \square

We are ready to give a proof of Lemma 12.10, and consequently Proposition 12.3 for $j = i$.

Proof of Lemma 12.10. Since $1 \leq i \leq n - 1$, the cases (3) and (4) occur in Lemma 12.13. Hence $M_{II'} = 0$ implies that

$$a \ell B = \frac{1}{2} a B \pm b = 0,$$

or equivalently $a = b = 0$ or $b = B = 0$. Since $(A, B) \neq (0, 0)$, the case $b = B = 0$ does not occur. The condition $a = 0$ implies $\lambda + \nu = n$, and the condition $b = 0$ gives the ratio of (A, B) as stated in (iii). Therefore the implication (ii) \Rightarrow (iii) is proved. The converse statement (iii) \Rightarrow (ii) is clear from Lemma 12.13. \square

12.6. Solving the F-system when $j = i - 1$. The case $j = i - 1$ is similar to the case $j = i + 1$.

According to Lemma 12.4, we may assume $\nu - \lambda = 2\ell + 1$ for some $\ell \in \mathbb{N}$ and $\beta \equiv \alpha + 1 \pmod{2}$. We set $\psi := Q_n^\ell H_{i \rightarrow i-1}^{(1)}$. Then we have:

Lemma 12.14. *Suppose $1 \leq i \leq n$. Then the following two conditions are equivalent:*

- (i) $\widehat{d\pi_{(i,\lambda)}^*}(N_1^+) \psi = 0$.
- (ii) $(\lambda, \nu) = (n - i, n - i + 1)$ and $\ell = 0$.

The proof of Lemma 12.14 goes similarly to that of Lemma 12.5 in the case $j = i + 1$, and so we omit it. Alternatively, Lemma 12.14 follows from Lemma 12.5. In fact,

$$\mathrm{Diff}_G(I(i, \lambda)_\alpha, I(j, \nu)_\beta) \simeq \mathrm{Diff}_G(I(n - i, \lambda)_\alpha, I(n - j, \nu)_\beta)$$
$$Sol(\mathbf{n}_+; \sigma_{\lambda, \alpha}^{(i)}, \sigma_{\nu, \beta}^{(j)}) \simeq Sol(\mathbf{n}_+; \sigma_{\lambda, \alpha}^{(n-i)}, \sigma_{\nu, \beta}^{(n-j)}).$$

12.7. Proof of Theorem 12.1. In this section, we deduce Theorem 12.1 (conformal representations) from Theorem 12.2 (principal series representations), as we did in Chapter 11 for symmetry breaking operators. Needless to say, the case ($G = G'$) in this section is much simpler than the case ($G \neq G'$) in the previous chapter. We recall from Proposition 2.3 that there are the natural isomorphisms as G -modules:

We note that there are two geometric models of the same principal series representations $I(i, \lambda)_i$. We translate the four cases in Theorem 12.2 in terms of (12.6).

We take $\alpha \equiv i$, and $\beta \equiv i + 1 \pmod{2}$. Then we have

Case 2. $j = i - 1$, $(\lambda, \nu) = (n - i, n - i + 1)$.

We take $\alpha \equiv i$ and $\beta \equiv i - 1 \pmod{2}$. Then we have

$$\begin{aligned} I(i, n-i)_i &\simeq \varpi_{n-2i,0}^{(i)} \simeq \varpi_{0,1}^{(n-i)}, \\ I(i-1, n-i+1)_{i-1} &\simeq \varpi_{n-2i+2,0}^{(i-1)} \simeq \varpi_{0,1}^{(n-i+1)}. \end{aligned}$$

Then the intertwining operators assured in Theorem 12.2 in Cases 1 and 2 are given in the following four arrows (after switching i and $n - i$ in Case 2):

$$\begin{array}{ccc}
\mathcal{E}^i(S^n)_{0,\delta} & \overset{\text{---}}{\underset{*}{\rightrightarrows}} & \mathcal{E}^{n-i}(S^n)_{2i-n,\delta+1} \\
\downarrow & \swarrow \quad \searrow & \downarrow \\
\mathcal{E}^{i+1}(S^n)_{0,\delta} & \overset{\text{---}}{\underset{*}{\rightrightarrows}} & \mathcal{E}^{n-i-1}(S^n)_{2i+2-n,\delta+1}
\end{array}$$

The vertical arrows are scalar multiples of d and d^* (see Proposition 12.3, and the horizontal dotted arrows are given by Hodge star operators. This explains the four Cases b, *b, c, and *c in Theorem 12.1.

Case 3. $j = i$, $\lambda + \nu = n$, $\nu - \lambda \in 2\mathbb{N}_+$.

We set $\ell := \frac{1}{2}(\nu - \lambda)$ so that $\lambda = \frac{n}{2} - \ell$ and $\nu = \frac{n}{2} + \ell$. Then we have isomorphisms as G -modules:

$$\begin{aligned} I\left(i, \frac{n}{2} - \ell\right)_i &\simeq \varpi_{\frac{n}{2} - \ell - i, 0}^{(i)} \simeq \varpi_{-\frac{n}{2} - \ell + i, 1}^{(n-i)}, \\ I\left(i, \frac{n}{2} + \ell\right)_i &\simeq \varpi_{\frac{n}{2} + \ell - i, 0}^{(i)} \simeq \varpi_{-\frac{n}{2} + \ell + i, 1}^{(n-i)}. \end{aligned}$$

This yields Cases d and *d in Theorem 12.1. Case 4 yields Cases a and *a. Thus we have listed all possible cases, and the proof of Theorem 12.1 is completed.

12.8. Hodge star operator and Branson's operator $\mathcal{T}_{2\ell}^{(i)}$. By the multiplicity-freeness result,

$$\dim \text{Diff}_{O(n+1,1)}(\mathcal{E}^i(S^n)_{u,\delta}, \mathcal{E}^j(S^n)_{v,\varepsilon}) \leq 1$$

in Theorem 12.1, we know *a priori* that the composition $* \circ \mathcal{T}_{2\ell}^{(i)} \circ *^{-1}$ of conformally covariant operators is proportional to $\mathcal{T}_{2\ell}^{(n-i)}$. To be explicit, we have the following proposition.

Proposition 12.15. *Let $0 \leq i \leq n$ and $\ell \in \mathbb{N}_+$. We put $u := \frac{n}{2} - i - \ell$, $v := \frac{n}{2} - i + \ell$. Then the following diagram commutes for any $\delta \in \mathbb{Z}/2\mathbb{Z}$.*

$$\begin{array}{ccc} \mathcal{E}^i(S^n)_{u,\delta} & \xrightarrow{\mathcal{T}_{2\ell}^{(i)}} & \mathcal{E}^i(S^n)_{v,\delta} \\ \downarrow * & & \downarrow * \\ \mathcal{E}^{n-i}(S^n)_{-v,\delta+1} & \xrightarrow{-\mathcal{T}_{2\ell}^{(n-i)}} & \mathcal{E}^{n-i}(S^n)_{-v+2\ell,\delta+1}. \end{array}$$

Proof. The vertical isomorphisms are given by Proposition 8.3 because $u - n + 2i = -v$. It then follows from Lemma 8.1 and (12.1) that

$$\mathcal{T}_{2\ell}^{(n-i)} = - *_{\mathbb{R}^n} \circ \mathcal{T}_{2\ell}^{(i)} \circ (*_{\mathbb{R}^n})^{-1}$$

in the flat coordinates. Hence the proposition follows from Lemma 8.2 (see also (11.6)). \square

13. MATRIX-VALUED FACTORIZATION IDENTITIES

Differential symmetry breaking operators may be expressed as a composition of two equivariant differential operators for some special values of the parameters. Such formulæ in the *scalar case* are called “factorization identities” in [11] or “functional identities for symmetry breaking operators” in [22].

In this chapter we establish “factorization identities” for conformally covariant differential operators on differential forms. A number of results have been known in the scalar case [11, 15, 19, 22], however, what we treat here is *matrix-valued* factorization identities. The setting we consider is $(X, Y) = (S^n, S^{n-1})$ and $(G, G') = (O(n+1, 1), O(n, 1))$ as before, and T_X (respectively, T_Y) is a G - (respectively, G' -) intertwining operator in the following diagrams:

$$\begin{array}{ccc}
 \mathcal{E}^i(X)_{u,\delta} & \xrightarrow{D_{u,a}^{i \rightarrow j}} & \mathcal{E}^j(Y)_{v,\varepsilon} \\
 \downarrow T_X & \searrow & \downarrow T_Y \\
 \mathcal{E}^{i'}(X)_{u',\delta'} & \xrightarrow{D_{u',b}^{i' \rightarrow j}} & \mathcal{E}^{j'}(Y)_{v',\varepsilon'} \\
 & \nearrow D_{u,c}^{i \rightarrow j'} &
 \end{array}$$

In Theorem 1.1, we have classified all the parameters for which there exist nonzero differential symmetry breaking operators, and have proved a multiplicity-one theorem. This guarantees the following “factorization identities” for some $p, q \in \mathbb{C}$:

$$\begin{aligned}
 D_{u',b}^{i' \rightarrow j} \circ T_X &= p D_{u,a}^{i \rightarrow j}, \\
 T_Y \circ D_{u,a}^{i \rightarrow j} &= q D_{u,c}^{i \rightarrow j'}.
 \end{aligned}$$

Explicit generators for symmetry breaking operators $D_{u,a}^{i \rightarrow j}$ are given in Theorems 1.5-1.8, whereas nontrivial G - or G' -intertwining operators T_X or T_Y are classified in Theorem 12.1. In this chapter, we shall consider all possible combinations of these operators under the parity condition

$$\delta \equiv \delta' \equiv \varepsilon \pmod{2} \quad \text{or} \quad \delta \equiv \varepsilon \equiv \varepsilon' \pmod{2},$$

and determine factorization identities. The explicit formulæ are given in Theorem 13.1 for $T_X = \mathcal{T}_{2\ell}^{(i)}$ (Branson’s operator), in Theorem 13.2 for $T_Y = \mathcal{T}_{2\ell}'^{(j)}$, in Theorem 13.3 for $T_X = d$ or d^* , and in Theorem 13.4 for $T_Y = d$ or d^* . Factorization identities for the other parity case are derived easily from the same parity case by using the Hodge star operators.

In Section 13.1, we summarize these factorization identities in terms of the unnormalized symmetry breaking operators $\mathcal{D}_{u,a}^{i \rightarrow j}$ rather than the renormalized ones $\widetilde{\mathcal{D}}_{u,a}^{i \rightarrow j}$,

because the formulæ take a simpler form. Factorization identities for the renormalized operators will be discussed in Section 13.8 with focus on the exact condition when the composition of two nonzero operators vanishes.

13.1. Matrix-valued factorization identities. This section summarizes the factorization identities for unnormalized operators for $(X, Y) = (S^n, S^{n-1})$ with the same parity of $\delta, \delta', \varepsilon$, and ε' .

We begin with the case $T_X = \mathcal{T}_{2\ell}^{(i)}$ or $T_Y = \mathcal{T}_{2\ell}'^{(j)}$, where we recall from Theorem 12.1 that for $\ell \in \mathbb{N}_+$ the G - and G' -intertwining operators (Branson's operators)

$$\begin{aligned} \mathcal{T}_{2\ell}^{(i)} &: \mathcal{E}^i(S^n)_{\frac{n}{2}-i-\ell} \longrightarrow \mathcal{E}^i(S^n)_{\frac{n}{2}-i+\ell}, \\ \mathcal{T}_{2\ell}'^{(j)} &: \mathcal{E}^j(S^{n-1})_{\frac{n-1}{2}-j-\ell} \longrightarrow \mathcal{E}^j(S^{n-1})_{\frac{n-1}{2}-j+\ell}. \end{aligned}$$

Let $\delta \equiv a + i + j \pmod{2}$. Here are basic diagrams:

$$\begin{array}{ccc} \mathcal{E}^i(S^n)_{\frac{n}{2}-i-\ell, \delta} & \xrightarrow{\mathcal{D}_{\frac{n}{2}-i-\ell, a+2\ell}^{i \rightarrow j}} & \mathcal{E}^j(S^{n-1})_{\frac{n-1}{2}-j-\ell, \delta} \\ \mathcal{T}_{2\ell}^{(i)} \downarrow & & \downarrow \mathcal{T}_{2\ell}'^{(j)} \\ \mathcal{E}^i(S^n)_{\frac{n}{2}-i+\ell, \delta} & \xrightarrow{\mathcal{D}_{\frac{n}{2}-i+\ell, a}^{i \rightarrow j}} & \mathcal{E}^j(S^{n-1})_{\frac{n-1}{2}-j+\ell, \delta} \end{array}$$

As in (1.13), we define a positive integer $K_{\ell, a}$ by $K_{\ell, a} = \prod_{k=1}^{\ell} \left(\left[\frac{a}{2} \right] + k \right)$ for $\ell \in \mathbb{N}_+$ and $a \in \mathbb{N}$. Then we have:

Theorem 13.1. *Suppose $0 \leq i \leq n, a \in \mathbb{N}$ and $\ell \in \mathbb{N}_+$. We set $u := \frac{n}{2} - i - \ell$. Then*

$$\begin{aligned} (1) \quad \mathcal{D}_{u+2\ell, a}^{i \rightarrow i-1} \circ \mathcal{T}_{2\ell}^{(i)} &= - \left(\frac{n}{2} - i - \ell \right) K_{\ell, a} \mathcal{D}_{u, a+2\ell}^{i \rightarrow i-1} \quad \text{if } i \neq 0. \\ (2) \quad \mathcal{D}_{u+2\ell, a}^{i \rightarrow i} \circ \mathcal{T}_{2\ell}^{(i)} &= - \left(\frac{n}{2} - i + \ell \right) K_{\ell, a} \mathcal{D}_{u, a+2\ell}^{i \rightarrow i} \quad \text{if } i \neq n. \end{aligned}$$

Theorem 13.2. *Suppose $0 \leq i \leq n, a \in \mathbb{N}$ and $\ell \in \mathbb{N}_+$. We set $u := \frac{n-1}{2} - i - \ell - a$. Then*

$$\begin{aligned} (1) \quad \mathcal{T}_{2\ell}'^{(i-1)} \circ \mathcal{D}_{u, a}^{i \rightarrow i-1} &= - \left(\frac{n+1}{2} - i + \ell \right) K_{\ell, a} \mathcal{D}_{u, a+2\ell}^{i \rightarrow i-1} \quad \text{if } i \neq 0. \\ (2) \quad \mathcal{T}_{2\ell}'^{(i)} \circ \mathcal{D}_{u, a}^{i \rightarrow i} &= - \left(\frac{n-1}{2} - i - \ell \right) K_{\ell, a} \mathcal{D}_{u, a+2\ell}^{i \rightarrow i} \quad \text{if } i \neq n. \end{aligned}$$

We shall prove Theorems 13.1 and 13.2 in Sections 13.4 and 13.5, respectively. Next, we consider the case where $T_X = d$ or d^* . Here are basic diagrams:

$$\begin{array}{ccc}
\mathcal{E}^i(S^n)_{0,\delta} & \xrightarrow{\mathcal{D}_{0,a+1}^{i \rightarrow j}} & \mathcal{E}^j(S^{n-1})_{a+1+i-j,\delta} \\
\downarrow d & & \\
\mathcal{E}^{i+1}(S^n)_{0,\delta} & \xrightarrow{\mathcal{D}_{0,a}^{i+1 \rightarrow j}} & \mathcal{E}^j(S^{n-1})_{a+1+i-j,\delta} \\
(j = i, i+1) & &
\end{array}
\quad
\begin{array}{ccc}
\mathcal{E}^i(S^n)_{n-2i,\delta} & \xrightarrow{\mathcal{D}_{n-2i,a+1}^{i \rightarrow j}} & \mathcal{E}^j(S^{n-1})_{n-i-j+a+1,\delta} \\
\downarrow d^* & & \\
\mathcal{E}^{i-1}(S^n)_{n-2i+2,\delta} & \xrightarrow{\mathcal{D}_{n-2i+2,a}^{i-1 \rightarrow j}} & \mathcal{E}^j(S^{n-1})_{n-i-j+a+1,\delta} \\
(j = i-1, i-2) & &
\end{array}$$

In this setting, the condition on $\delta \in \mathbb{Z}/2\mathbb{Z}$ from Theorem 1.1 is listed in the theorem below.

Theorem 13.3. *For any $a \in \mathbb{N}$, we have*

- (1) $\mathcal{D}_{0,a}^{i+1 \rightarrow i} \circ d = \gamma(i+1 - \frac{n}{2}, a) \mathcal{D}_{0,a+1}^{i \rightarrow i}, \quad 0 \leq i \leq n-1, \quad \delta \equiv a+1 \pmod{2}.$
- (2) $\mathcal{D}_{0,a}^{i+1 \rightarrow i+1} \circ d = 0, \quad 0 \leq i \leq n-1, \quad \delta \equiv 0 \pmod{2}.$
- (3) $\mathcal{D}_{n-2i+2,a}^{i-1 \rightarrow i-1} \circ d^* = -\gamma(-i+1 + \frac{n}{2}, a) \mathcal{D}_{n-2i,a+1}^{i \rightarrow i-1}, \quad 1 \leq i \leq n, \quad \delta \equiv a \pmod{2}.$
- (4) $\mathcal{D}_{n-2i+2,a}^{i-1 \rightarrow i-2} \circ d^* = 0, \quad 2 \leq i \leq n, \quad \delta \equiv 1 \pmod{2}.$

Here we recall from (1.3) that $\gamma(\mu, a) = 1$ if a is odd; $= \mu + \frac{a}{2}$ if a is even.

The proof of Theorem 13.3 will be given in Section 13.6.

Finally, we consider the case where $T_Y = d$ or d^* . Here are basic diagrams:

$$\begin{array}{ccc}
\mathcal{E}^i(S^n)_{-a-i+j,\delta} & \xrightarrow{\mathcal{D}_{-a-i+j,a}^{i \rightarrow j}} & \mathcal{E}^j(S^{n-1})_{0,\delta} \\
& \searrow \mathcal{D}_{-a-i+j,a+1}^{i \rightarrow j+1} & \downarrow d \\
& & \mathcal{E}^{j+1}(S^{n-1})_{0,\delta}, \\
(j = i-1, i) & &
\end{array}
\quad
\begin{array}{ccc}
\mathcal{E}^i(S^n)_{n-i-j-a-1,\delta} & \xrightarrow{\mathcal{D}_{n-i-j-a-1,a}^{i \rightarrow j}} & \mathcal{E}^j(S^{n-1})_{n-2j-1,\delta} \\
& \searrow \mathcal{D}_{n-i-j-a-1,a+1}^{i \rightarrow j-1} & \downarrow d^* \\
& & \mathcal{E}^{j-1}(S^{n-1})_{n-2j+1,\delta}. \\
(j = i-1, i) & &
\end{array}$$

In these two diagrams, $\delta \equiv a + i + j \pmod{2}$.

Theorem 13.4. *For any $a \in \mathbb{N}$, we have*

- (1) $d \circ \mathcal{D}_{-a-1,a}^{i \rightarrow i-1} = -\gamma\left(-a+i - \frac{n+1}{2}, a\right) \mathcal{D}_{-a-1,a+1}^{i \rightarrow i} \quad \text{for } 1 \leq i \leq n-1.$
- (2) $d \circ \mathcal{D}_{-a,a}^{i \rightarrow i} = 0 \quad \text{for } 0 \leq i \leq n-2.$
- (3) $d^* \circ \mathcal{D}_{n-2i-a-1,a}^{i \rightarrow i} = -\gamma\left(-a-i + \frac{n-1}{2}, a\right) \mathcal{D}_{n-2i-a-1,a+1}^{i \rightarrow i-1} \quad \text{for } 1 \leq i \leq n.$
- (4) $d^* \circ \mathcal{D}_{n-2i-a,a}^{i \rightarrow i-1} = 0 \quad \text{for } 2 \leq i \leq n.$

Here we consider only the case where $a = 0$ if $1 \leq i \leq n-1$ in (2), or if $2 \leq i \leq n-1$ in (4). The proof of Theorem 13.4 will be given in Section 13.7.

Remark 13.5. Theorem 13.3 (2) for $a \neq 0$ reflects Theorem 1.1 which asserts that there is no nonzero conformally equivariant symmetry breaking operators $\mathcal{E}^i(S^n) \rightarrow \mathcal{E}^{i+1}(S^{n-1})$ other than the obvious one $\text{Rest}_{S^n} \circ d$ (i.e. $a = 0$ case) up to scalar if $i \neq 0$. On the other hand, we recall from Proposition 1.4 that the differential symmetry breaking operators $\mathcal{D}_{u,a}^{i \rightarrow j}$ may vanish for specific parameters, for instance,

$$\mathcal{D}_{-a,a}^{i \rightarrow i} = 0 \quad \text{if } i = 0 \quad \text{or} \quad a = 0.$$

In such cases, the fomulæ like Theorem 13.3 (2) for $a = 0$ or Theorem 13.4 (2) for $i = 0$ or $a = 0$ are trivial, however, by using the renormalized operators $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i}$ and $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i+1}$ given in (1.10) and (1.11), we have the following nontrivial factorization identities:

$$\begin{aligned} \tilde{\mathcal{D}}_{0,0}^{i+1,i+1} \circ d &= \tilde{\mathcal{D}}_{0,0}^{i \rightarrow i+1}, \\ d \circ \tilde{\mathcal{D}}_{-a,a}^{0 \rightarrow 0} &= \tilde{\mathcal{D}}_{-a,a+1}^{0 \rightarrow 1} \quad \text{for all } a \in \mathbb{N}, \\ d \circ \tilde{\mathcal{D}}_{0,0}^{i \rightarrow i} &= \tilde{\mathcal{D}}_{0,1}^{i \rightarrow i}. \end{aligned}$$

We shall discuss in Section 13.8 the factorization identities for renormalized symmetry breaking operators of this kind corresponding to Theorem 13.3 (2) and (4), and Theorem 13.4 (2) and (4) among others. We also determine the vanishing condition of the composition of two nonzero operators in detail.

As an immediate corollary of Theorem 13.4, we can tell exactly when the image of the symmetry breaking operator $\mathcal{D}_{-a-1,a}^{i \rightarrow i-1}$ consists of closed forms on the submanifold S^{n-1} .

Corollary 13.6.

(1) Assume that n is odd and a is even with $0 \leq a \leq n-1$. We set

$$(13.1) \quad i := \frac{1}{2}(a + n + 1).$$

Then $\frac{n+1}{2} \leq i \leq n$ and $\mathcal{D}_{-a-1,a}^{i \rightarrow i-1} \omega$ is a closed i -form on S^{n-1} for any i -form ω on S^n .

(2) Conversely, suppose $1 \leq i \leq n-1$. If n is even or if $a \neq 2i - n - 1$, then there exists $\omega \in \mathcal{E}^i(S^n)$ such that $\mathcal{D}_{-a-1,a}^{i \rightarrow i-1} \omega$ is not a closed form on S^{n-1} .

Proof. For $i = n$, the $(n-1)$ -form $\mathcal{D}_{u,a}^{n \rightarrow n-1} \omega$ is automatically closed for any $\omega \in \mathcal{E}^n(S^n)$. Suppose $1 \leq i \leq n-1$ and $a \in \mathbb{N}$. By Theorem 13.4 (1), $d \circ \mathcal{D}_{-a-1,a}^{i \rightarrow i-1} = 0$ if and only if $\gamma(-a - i + \frac{n+1}{2}, a) = 0$. By the definition (1.3) of $\gamma(\mu, a)$, this happens exactly when a is even and $i = \frac{1}{2}(a + n + 1)$. This forces n to be even. We also note

that $0 \leq a$ is equivalent to $\frac{1}{2}(n+1) \leq i$, and $a \leq n-1$ is equivalent to $i \leq n$. Hence the corollary follows. \square

13.2. Proof of Theorem 13.1 (1). We shall work with the flat coordinates. Theorem 13.1 (1) will be shown by the following proposition.

Proposition 13.7. *There exist scalar-valued differential operators P, Q, R, P', Q' and R' on \mathbb{R}^n satisfying the following three conditions:*

$$(13.2) \quad \begin{aligned} \mathcal{D}_{\frac{n}{2}-i-\ell, a+2\ell}^{i \rightarrow i-1} &= \text{Rest}_{x_n=0} \circ \left(P d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} + Q d_{\mathbb{R}^n}^* + R \iota_{\frac{\partial}{\partial x_n}} \right), \\ \mathcal{D}_{\frac{n}{2}-i+\ell, a}^{i \rightarrow i-1} \circ \mathcal{T}_{2\ell}^{(i)} &= \text{Rest}_{x_n=0} \circ \left(P' d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} + Q' d_{\mathbb{R}^n}^* + R' \iota_{\frac{\partial}{\partial x_n}} \right), \\ P &= c_i(\ell) P', \quad Q = c_i(\ell) Q', \quad R = c_i(\ell) R', \end{aligned}$$

where $c_i(\ell)$ is defined by

$$c_i(\ell) := -\left(\frac{n}{2} - i - \ell\right) K_{\ell, a} \left(= -\left(\frac{n}{2} - i - \ell\right) \prod_{k=1}^{\ell} \left(\left[\frac{a}{2}\right] + k\right) \right).$$

The rest of this section is devoted to the proof of Proposition 13.7.

We take

$$\begin{aligned} P &= -\mathcal{D}_{a+2\ell-2}^{-\ell+\frac{3}{2}}, \\ Q &= -\gamma(-\ell + \frac{1}{2}, a+2\ell) \mathcal{D}_{a+2\ell-1}^{-\ell+\frac{3}{2}}, \\ R &= -\frac{1}{2} \left(\frac{n}{2} - i + \ell\right) \mathcal{D}_{a+2\ell}^{-\ell+\frac{1}{2}}, \end{aligned}$$

according to the formula (1.4) of $\mathcal{D}_{u,a}^{i \rightarrow i-1}$. In order to find P', Q' , and R' , we use:

Lemma 13.8. *Suppose A, B, C, p and q are (scalar-valued) differential operators on \mathbb{R}^n with constant coefficients. We set*

$$P' := -Ap\Delta_{\mathbb{R}^n} + Cp + Aq, \quad Q' := -Bp\Delta_{\mathbb{R}^n} + Cp\frac{\partial}{\partial x_n} + Bq, \quad R' := Cq.$$

Then,

$$\left(A d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} + B d_{\mathbb{R}^n}^* + C \iota_{\frac{\partial}{\partial x_n}} \right) \circ (p d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* + q) = P' d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} + Q' d_{\mathbb{R}^n}^* + R' \iota_{\frac{\partial}{\partial x_n}}.$$

Proof. This is an easy consequence of the commutation relations among the operators $d_{\mathbb{R}^n}$, $d_{\mathbb{R}^n}^*$, and $\iota_{\frac{\partial}{\partial x_n}}$ given in Lemma 8.14 together with $(d_{\mathbb{R}^n}^*)^2 = 0$ and $d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* + d_{\mathbb{R}^n}^* d_{\mathbb{R}^n} = -\Delta_{\mathbb{R}^n}$. \square

Lemma 13.9. *The equation (13.2) holds if we take*

$$\begin{aligned} P' &= \left(\frac{n}{2} - i - \ell\right) \left(\ell + \left[\frac{a}{2}\right]\right) \mathcal{D}_a^{\ell-\frac{1}{2}} \Delta_{\mathbb{R}^n}^{\ell-1}, \\ Q' &= \frac{\left(\frac{n}{2} - i - \ell\right) (a+1)(a+2\ell)}{4\gamma\left(\ell + \frac{1}{2}, a-1\right)} \mathcal{D}_{a+1}^{\ell-\frac{1}{2}} \Delta_{\mathbb{R}^n}^{\ell-1}, \\ R' &= \frac{1}{2} \left(\frac{n}{2} - i - \ell\right) \left(\frac{n}{2} - i + \ell\right) \mathcal{D}_a^{\ell+\frac{1}{2}} \Delta_{\mathbb{R}^n}^{\ell}. \end{aligned}$$

Proof. By Theorems 1.5 and 12.1,

$$\begin{aligned} \mathcal{D}_{\frac{n}{2}-i+\ell,a}^{i \rightarrow i-1} &= \text{Rest}_{x_n=0} \circ \left(A d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* + B d_{\mathbb{R}^n}^* + C \iota_{\frac{\partial}{\partial x_n}} \right), \\ \mathcal{T}_{2\ell}^{(i)} &= p d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* + q, \end{aligned}$$

if we set

$$A = -\mathcal{D}_{a-2}^{\ell+\frac{3}{2}}, \quad B = -\gamma\left(\ell + \frac{1}{2}, a\right) \mathcal{D}_{a-1}^{\ell+\frac{3}{2}}, \quad C = -\frac{1}{2} \left(\frac{n}{2} - i - \ell\right) \mathcal{D}_a^{\ell+\frac{1}{2}}, \quad p = -2\ell, \quad q = -\left(\frac{n}{2} - i + \ell\right).$$

Applying Lemma 13.8, we get (13.2) if we set

$$\begin{aligned} P' &= \left(\frac{n}{2} - i - \ell\right) \left(\mathcal{D}_{a-2}^{\ell+\frac{3}{2}} \Delta_{\mathbb{R}^n} + \ell \mathcal{D}_a^{\ell+\frac{1}{2}}\right) \Delta_{\mathbb{R}^n}^{\ell-1}, \\ Q' &= \left(\frac{n}{2} - i - \ell\right) \left(\gamma\left(\ell + \frac{1}{2}, a\right) \mathcal{D}_{a-1}^{\ell+\frac{3}{2}} \Delta_{\mathbb{R}^n} + \ell \mathcal{D}_a^{\ell+\frac{1}{2}} \frac{\partial}{\partial x_n}\right) \Delta_{\mathbb{R}^n}^{\ell-1}, \\ R' &= \frac{1}{2} \left(\frac{n}{2} - i - \ell\right) \left(\frac{n}{2} - i + \ell\right) \mathcal{D}_a^{\ell+\frac{1}{2}} \Delta_{\mathbb{R}^n}^{\ell}. \end{aligned}$$

Then the lemma follows from the three-term relations (9.8) for P' , (9.9) for Q' with $\nu = \ell + \frac{1}{2}$. \square

We are ready to give a proof of Proposition 13.7.

Proof of Proposition 13.7. The assertions $P = c_i(\ell)P'$, $Q = c_i(\ell)Q'$, and $R = c_i(\ell)R'$ are now reduced to the factorization identities for scalar-valued differential operators (Juhl's operators) which were proved in Lemma 9.4 (1), (2), and Proposition 9.3 (1), respectively. \square

Thus the proof of Theorem 13.1 (1) is completed.

13.3. Proof of Theorem 13.1 (2). We deduce the second statement of Theorem 13.1 from the first one by using the Hodge star operator. By Theorem 13.1 (1) with $\tilde{i} := n - i$, we have

$$(13.3) \quad \mathcal{D}_{\frac{n}{2}-\tilde{i}+\ell,a}^{\tilde{i} \rightarrow \tilde{i}-1} \circ \mathcal{T}_{2\ell}^{(\tilde{i})} = -\left(\frac{n}{2} - \tilde{i} - \ell\right) K_{\ell,a} \mathcal{D}_{\frac{n}{2}-\tilde{i}-\ell,a+2\ell}^{\tilde{i} \rightarrow \tilde{i}-1}.$$

We recall from Proposition 12.15

$$*\mathbb{R}^n \circ \mathcal{T}_{2\ell}^{(\tilde{i})} \circ (*\mathbb{R}^n)^{-1} = -\mathcal{T}_{2\ell}^{(\tilde{i})},$$

and from (1.8) with $\tilde{i} := n - i$

$$\begin{aligned} \mathcal{D}_{\frac{n}{2}-i+\ell,a}^{i \rightarrow i} &= (-1)^{n-1} *_{\mathbb{R}^{n-1}} \circ \mathcal{D}_{\frac{n}{2}-\tilde{i}+\ell,a}^{\tilde{i} \rightarrow \tilde{i}-1} \circ (*\mathbb{R}^n)^{-1}, \\ \mathcal{D}_{\frac{n}{2}-i+\ell,a+2\ell}^{i \rightarrow i} &= (-1)^{n-1} *_{\mathbb{R}^{n-1}} \circ \mathcal{D}_{\frac{n}{2}-\tilde{i}+\ell,a+2\ell}^{\tilde{i} \rightarrow \tilde{i}-1} \circ (*\mathbb{R}^n)^{-1}. \end{aligned}$$

Then Theorem 13.1 (2) follows from (13.3) by applying $*_{\mathbb{R}^{n-1}}$ from the left and $(*\mathbb{R}^n)^{-1}$ from the right.

13.4. Proof of Theorem 13.2 (1). This section gives a proof of Theorem 13.2 (1). As in the proof of Theorem 13.1 (2), we shall reduce the proof to an analogous identity for the scalar-valued case (Proposition 9.3 (2)). For this, we need some computation for matrix-valued differential operators that are stated in the lemmas below.

Lemma 13.10. *Suppose that p, q, r, A, B and C are (scalar-valued) differential operators with constant coefficients on \mathbb{R}^n . We set*

$$R := -pA\Delta_{\mathbb{R}^{n-1}} - pA\frac{\partial^2}{\partial x_n^2} + pC + rA - qB + qA\frac{\partial}{\partial x_n}.$$

Then the following identity holds:

$$(pd_{\mathbb{R}^n}d_{\mathbb{R}^n}^* + qd_{\mathbb{R}^n}\iota_{\frac{\partial}{\partial x_n}} + r) \circ (Ad_{\mathbb{R}^n}d_{\mathbb{R}^n}^*\iota_{\frac{\partial}{\partial x_n}} + Bd_{\mathbb{R}^n}^* + C\iota_{\frac{\partial}{\partial x_n}}) = Rd_{\mathbb{R}^n}d_{\mathbb{R}^n}^*\iota_{\frac{\partial}{\partial x_n}} + rBd_{\mathbb{R}^n}^* + rC\iota_{\frac{\partial}{\partial x_n}}.$$

Proof. Direct computation by Lemma 8.14, as in the proof of Lemma 13.8. \square

Lemma 13.11. *Let p, q , and r be (scalar-valued) differential operators of 0th, first, and second order, respectively, given by*

$$p = -2\ell, \quad q = -2\ell\frac{\partial}{\partial x_n}, \quad r = -\left(\frac{n+1}{2} + \ell - i\right)\Delta_{\mathbb{R}^{n-1}}.$$

Then,

$$\mathcal{T}_{2\ell}^{(i-1)} \circ \text{Rest}_{x_n=0} = \text{Rest}_{x_n=0} \circ (pd_{\mathbb{R}^n}d_{\mathbb{R}^n}^* + qd_{\mathbb{R}^n}\iota_{\frac{\partial}{\partial x_n}} + r).$$

Proof. The identity follows from the commutation relations among $\text{Rest}_{x_n=0}$, $d_{\mathbb{R}^n}d_{\mathbb{R}^n}^*$, and $d_{\mathbb{R}^n}^*d_{\mathbb{R}^n}$ given in Lemma 8.15 (3) and (4). \square

Let $u := \frac{n-1}{2} - i - a - \ell$. Then by the formula (1.4) of $\mathcal{D}_{u,a}^{i \rightarrow i-1}$, we have

$$\mathcal{D}_{u,a}^{i \rightarrow i-1} = \text{Rest}_{x_n=0} \circ (Ad_{\mathbb{R}^n}d_{\mathbb{R}^n}^*\iota_{\frac{\partial}{\partial x_n}} + Bd_{\mathbb{R}^n}^* + C\iota_{\frac{\partial}{\partial x_n}}),$$

where

(13.4)

$$A := -\mathcal{D}_{a-2}^{-a-\ell+1}, \quad B := -\gamma(-a-\ell, a)\mathcal{D}_{a-1}^{-a-\ell+1}, \quad C := -\frac{1}{2} \left(\frac{n+1}{2} - i + a + \ell \right) \mathcal{D}_a^{-a-\ell}.$$

Then the composition

$$\mathcal{T}'_{2\ell}^{(i-1)} \circ \mathcal{D}_{u,a}^{i \rightarrow i-1} = \left(\mathcal{T}'_{2\ell}^{(i-1)} \circ \text{Rest}_{x_n=0} \right) \circ \left(Ad_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} + B d_{\mathbb{R}^n}^* + C \iota_{\frac{\partial}{\partial x_n}} \right)$$

can be computed explicitly as follows.

Lemma 13.12.

$$\begin{aligned} \mathcal{T}'_{2\ell}^{(i-1)} \circ \mathcal{D}_{u,a}^{i \rightarrow i-1} &= \left(\frac{n+1}{2} + \ell - i \right) \text{Rest}_{x_n=0} \circ \left(\left(\ell + \left\lceil \frac{a}{2} \right\rceil \right) \mathcal{D}_a^{-a-\ell+1} \Delta_{\mathbb{R}^{n-1}}^{\ell-1} d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} \right. \\ &\quad \left. + \gamma(-a-\ell, a) \mathcal{D}_a^{-a-\ell+1} \Delta_{\mathbb{R}^{n-1}}^{\ell} d_{\mathbb{R}^n}^* + \frac{1}{2} \left(\frac{n+1}{2} - i + a + \ell \right) \mathcal{D}_a^{-a-\ell} \Delta_{\mathbb{R}^{n-1}}^{\ell} \iota_{\frac{\partial}{\partial x_n}} \right). \end{aligned}$$

Proof. The first two terms are derived directly from Lemmas 13.10 and 13.11. For the third term, we use three-term relations among \mathcal{D}_ℓ^λ 's that were studied in Chapter 9. What we actually need is the claim below. \square

Claim 13.13. Suppose p, q and r are given as in Lemma 13.11 and A, B and C by (13.4). Then the differential operator R in Lemma 13.10 amounts to

$$\left(\frac{n+1}{2} + \ell - i \right) \left(\ell + \left\lceil \frac{a}{2} \right\rceil \right) \mathcal{D}_a^{-a-\ell+1}.$$

Proof of Claim 13.13. A direct computation shows

$$\begin{aligned} R &= \left(\frac{n+1}{2} - \ell - i \right) \mathcal{D}_{a-2}^{-a-\ell+1} \Delta_{\mathbb{R}^{n-1}} + \left(\frac{n+1}{2} + a + \ell - i \right) \ell \mathcal{D}_a^{-a-\ell} \\ &\quad - 2\ell \gamma(-a-\ell, a) \mathcal{D}_{a-1}^{-a-\ell+1} \frac{\partial}{\partial x_n}. \end{aligned}$$

Applying the three-term relation (9.6) to $\mathcal{D}_{a-1}^{-a-\ell+1} \frac{\partial}{\partial x_n}$ in the last term, we see

$$R = \left(\frac{n+1}{2} + \ell - i \right) \left(\mathcal{D}_{a-2}^{-a-\ell+1} \Delta_{\mathbb{R}^{n-1}} + \ell \mathcal{D}_a^{-a-\ell} \right).$$

Finally, by the three-term relation (9.10) with $\mu = -a - \ell$, we get the claim. \square

We are ready to complete the proof of Theorem 13.2 (1). By the definition (1.4) of $\mathcal{D}_{u,b}^{i \rightarrow i-1}$ again,

$$\begin{aligned} &\mathcal{D}_{u,a+2\ell}^{i \rightarrow i-1} \\ &= \text{Rest}_{x_n=0} \circ \left(-\mathcal{D}_{a+2\ell-2}^{-a-\ell+1} d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} - \gamma(-a, a) \mathcal{D}_{a+2\ell-1}^{-a-\ell+1} - \frac{1}{2} \left(\frac{n+1}{2} - i + a + \ell \right) \mathcal{D}_{a+2\ell}^{-a-\ell} \iota_{\frac{\partial}{\partial x_n}} \right). \end{aligned}$$

Substituting the formulæ for the (scalar) differential operators \mathcal{D}_ℓ^λ from Lemma 9.4, we have

$$\begin{aligned} \mathcal{D}_{u,a+2\ell}^{i \rightarrow i-1} &= -K_{\ell,a}^{-1} \text{Rest}_{x_n=0} \circ \left(\left(\ell + \left\lfloor \frac{a}{2} \right\rfloor \right) \mathcal{D}_a^{-a-\ell+1} \Delta_{\mathbb{R}^{n-1}}^{\ell-1} d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} \right. \\ &\quad \left. + \gamma(-a-\ell, a) \mathcal{D}_{a-1}^{-a-\ell+1} \Delta_{\mathbb{R}^{n-1}}^\ell d_{\mathbb{R}^n}^* + \frac{1}{2} \left(\frac{n+1}{2} - i + a + \ell \right) \mathcal{D}_a^{-a-\ell} \Delta_{\mathbb{R}^{n-1}}^\ell \iota_{\frac{\partial}{\partial x_n}} \right). \end{aligned}$$

Comparing this with Lemma 13.12, we have completed the proof of Theorem 13.2 (1).

13.5. Proof of Theorem 13.2 (2). We deduce the second statement of Theorem 13.2 from the first one by using the Hodge operator $*$. By Theorem 13.2 (1) with $\tilde{i} := n - i$ and $\tilde{u} := \frac{n-1}{2} - \tilde{i} - a - \ell$, we have

$$(13.5) \quad \mathcal{T}'_{2\ell}^{(\tilde{i}-1)} \circ \mathcal{D}_{\tilde{u},a}^{\tilde{i} \rightarrow \tilde{i}-1} = - \left(\frac{n+1}{2} - \tilde{i} + \ell \right) K_{\ell,a} \mathcal{D}_{\tilde{u},a+2\ell}^{\tilde{i} \rightarrow \tilde{i}-1}.$$

We recall from Proposition 12.15

$$*_{\mathbb{R}^{n-1}} \circ \mathcal{T}'_{2\ell}^{(\tilde{i}-1)} \circ (*_{\mathbb{R}^{n-1}})^{-1} = -\mathcal{T}'_{2\ell}^{(\tilde{i})},$$

and from (1.6)

$$\begin{aligned} (-1)^{n-1} *_{\mathbb{R}^{n-1}} \circ \mathcal{D}_{\tilde{u},a}^{\tilde{i} \rightarrow \tilde{i}-1} \circ (*_{\mathbb{R}^n})^{-1} &= \mathcal{D}_{u,a}^{i \rightarrow i}, \\ (-1)^{n-1} *_{\mathbb{R}^{n-1}} \circ \mathcal{D}_{\tilde{u},a+2\ell}^{\tilde{i} \rightarrow \tilde{i}-1} \circ (*_{\mathbb{R}^n})^{-1} &= \mathcal{D}_{u,a+2\ell}^{i \rightarrow i}. \end{aligned}$$

Then Theorem 13.2 (2) follows by applying $*_{\mathbb{R}^{n-1}}$ from the left and $(*_{\mathbb{R}^n})^{-1}$ from the right to (13.5).

13.6. Proof of Theorem 13.3. In this section, we complete the proof of Theorem 13.3.

(1) We first show the following lemma:

Lemma 13.14. *Let $0 \leq i \leq n-1$ and $a \in \mathbb{N}$. We set $\mu := i - \frac{n-3}{2}$ and define the following scalar-valued differential operators:*

$$\begin{aligned} P' &:= -\mathcal{D}_{a-2}^{\mu+1} \frac{\partial}{\partial x_n} + \gamma(\mu, a) \mathcal{D}_{a-1}^{\mu+1}, \\ Q' &:= -\mathcal{D}_{a-2}^{\mu+1} \Delta_{\mathbb{R}^n} - \left(i + 1 - \frac{n}{2} \right) \mathcal{D}_a^\mu, \\ R' &:= \gamma(\mu, a) \mathcal{D}_{a-1}^{\mu+1} \Delta_{\mathbb{R}^n} + \left(i + 1 - \frac{n}{2} \right) \mathcal{D}_a^\mu \frac{\partial}{\partial x_n}. \end{aligned}$$

Then

$$\mathcal{D}_{0,a}^{i+1 \rightarrow i} \circ d_{\mathbb{R}^n} = \text{Rest}_{x_n=0} \circ \left(P' d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* + Q' d_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}} + R' \right).$$

Proof. The lemma follows from the formula (1.4) for $\mathcal{D}_{0,a}^{i+1 \rightarrow i}$ by (8.14) and Lemma 8.14 (1). \square

We observe that $i + 1 - \frac{n}{2} = \mu - \frac{1}{2}$. Then applying the three-term relations (9.7), (9.8), and (9.9) for \mathcal{D}_a^μ s, we see that the scalar-valued operators P' , Q' , and R' in Lemma 13.14 amounts to

$$\begin{aligned} P' &= \gamma(\mu - \frac{1}{2}, a) \mathcal{D}_{a-1}^\mu, \\ Q' &= - \left(\mu + \left[\frac{a}{2} \right] - \frac{1}{2} \right) \mathcal{D}_a^{\mu-1}, \\ R' &= \frac{1}{2}(a+1) \gamma(\mu - \frac{1}{2}, a) \mathcal{D}_{a+1}^{\mu-1}, \end{aligned}$$

respectively. In view of the formula (1.6) of $\mathcal{D}_{0,a+1}^{i \rightarrow i}$ and of the following elementary identity of $\gamma(\mu, a)$ (see (1.3))

$$\gamma(\mu - \frac{3}{2}, a+1) \gamma(\mu - \frac{1}{2}, a) = \mu + \left[\frac{a}{2} \right] - \frac{1}{2},$$

we obtain the first statement of Theorem 13.3.

(2) By the formula (1.7) for $\mathcal{D}_{0,a}^{i \rightarrow i}$, we get $\mathcal{D}_{0,a}^{i \rightarrow i} \circ d_{\mathbb{R}^n} = 0$ from Lemma 8.15 (1) and $d_{\mathbb{R}^n}^2 = 0$.

(3) By Theorem 13.3 (1) with i replaced by $n-i$, we have

$$\mathcal{D}_{0,a}^{n-i+1 \rightarrow n-i} \circ d_{\mathbb{R}^n} = \gamma(-i+1 + \frac{n}{2}, a) \mathcal{D}_{0,a+1}^{n-i \rightarrow n-i}.$$

Then we apply $*_{\mathbb{R}^{n-1}}$ from the left and $(*_{\mathbb{R}^n})^{-1}$ from the right to the above identity, and use the following identities

$$\begin{aligned} *_{\mathbb{R}^{n-1}} \circ \mathcal{D}_{0,a}^{n-i+1 \rightarrow n-i} \circ (*_{\mathbb{R}^n})^{-1} &= (-1)^{n-1} \mathcal{D}_{n-2i,a}^{i-1 \rightarrow i-1}, \\ *_{\mathbb{R}^n} \circ d_{\mathbb{R}^n} \circ (*_{\mathbb{R}^n})^{-1} &= (-1)^{n-i+1} d_{\mathbb{R}^n}^* \quad \text{on } \mathcal{E}^i(\mathbb{R}^n), \\ *_{\mathbb{R}^{n-1}} \circ \mathcal{D}_{0,a+1}^{n-i \rightarrow n-i} \circ (*_{\mathbb{R}^n})^{-1} &= (-1)^{i+1} \mathcal{D}_{n-2i,a}^{i \rightarrow i-1} \quad \text{on } \mathcal{E}^i(\mathbb{R}^n), \end{aligned}$$

as in the proof of Theorem 13.2 (2). Now the third statement of Theorem 13.3 follows.

(4) By the formula (1.4) for $\mathcal{D}_{n-2i+2,0}^{i-1 \rightarrow i-2}$, we get $\mathcal{D}_{n-2i+2,0}^{i-1 \rightarrow i-2} \circ d_{\mathbb{R}^n}^* = 0$ from Lemma 8.14 (2) and $(d_{\mathbb{R}^n}^*)^2 = 0$.

Thus Theorem 13.3 has been proved.

13.7. Proof of Theorem 13.4. In this section we complete the proof of Theorem 13.4.

Proof of Theorem 13.4 (1). We set

$$u := -a - 1, \quad \mu := u + i - \frac{n-1}{2}.$$

By the first expression of (1.4), we have

$$d_{\mathbb{R}^{n-1}} \circ \mathcal{D}_{-a-1,a}^{i \rightarrow i-1} = \text{Rest}_{x_n=0} \circ \left(-\gamma(\mu, a) \mathcal{D}_{a-1}^{\mu+1} d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* + \frac{1}{2} (u + 2i - n) \mathcal{D}_a^\mu d_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}} \right).$$

On the other hand, by the formula (1.6) of $\mathcal{D}_{u,a}^{i \rightarrow i}$, we have

$$\mathcal{D}_{-a-1,a+1}^{i \rightarrow i} = \text{Rest}_{x_n=0} \circ \left(\mathcal{D}_{a-1}^{\mu+1} d_{\mathbb{R}^n} d_{\mathbb{R}^n}^* - \gamma\left(\mu - \frac{1}{2}, a+1\right) \mathcal{D}_a^\mu d_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}} \right).$$

It follows from these formulæ that we get

$$(13.6) \quad d_{\mathbb{R}^{n-1}} \circ \mathcal{D}_{-a-1,a}^{i \rightarrow i-1} + \gamma(\mu, a) \mathcal{D}_{-a-1,a+1}^{i \rightarrow i} = \\ \text{Rest}_{x_n=0} \circ \left(\frac{1}{2} (u + 2i - n) - \gamma(\mu, a) \gamma\left(\mu - \frac{1}{2}, a+1\right) \right) \mathcal{D}_a^\mu d_{\mathbb{R}^n} \iota_{\frac{\partial}{\partial x_n}}.$$

By using the following elementary identity

$$\gamma(\mu, a) \gamma\left(\mu - \frac{1}{2}, a+1\right) = \mu + \frac{a}{2},$$

we see that the right-hand side of (13.6) vanishes. \square

Proof of Theorem 13.4 (2). It follows from the formula (1.6) of $\mathcal{D}_{u,a}^{i \rightarrow i}$ and Lemma 8.15 (1) that we have

$$d_{\mathbb{R}^{n-1}} \circ \mathcal{D}_{u,a}^{i \rightarrow i} = \frac{1}{2} (u + a) \text{Rest}_{x_n=0} \circ \mathcal{D}_a^\mu \circ d_{\mathbb{R}^n}.$$

Hence $d_{\mathbb{R}^{n-1}} \circ \mathcal{D}_{-a,a}^{i \rightarrow i} = 0$. \square

Proof of Theorem 13.4 (3). By Theorem 13.4 (1) with i replaced by $n-i$, we have

$$(13.7) \quad d_{\mathbb{R}^{n-1}} \circ \mathcal{D}_{-a-1,a}^{n-i \rightarrow n-i-1} = -\gamma\left(-a-i+\frac{n-1}{2}, a\right) \mathcal{D}_{-a-1,a+1}^{n-i \rightarrow n-i}.$$

We recall from (8.13)

$$*\mathbb{R}^{n-1} \circ d_{\mathbb{R}^{n-1}} \circ (*\mathbb{R}^{n-1})^{-1} = (-1)^{n-i} d_{\mathbb{R}^{n-1}}^* \quad \text{on } \mathcal{E}^i(\mathbb{R}^{n-1}),$$

and from (1.8)

$$\begin{aligned} *\mathbb{R}^{n-1} \circ \mathcal{D}_{-a-1,a}^{n-i \rightarrow n-i-1} \circ (*\mathbb{R}^n)^{-1} &= (-1)^{n-1} \mathcal{D}_{n-2i-a-1,a}^{i \rightarrow i}, \\ (*\mathbb{R}^{n-1})^{-1} \circ \mathcal{D}_{-a-1,a+1}^{n-i \rightarrow n-i} \circ *\mathbb{R}^n &= (-1)^{n-1} \mathcal{D}_{n-2i-a-1,a+1}^{i \rightarrow i-1}. \end{aligned}$$

Since $(*_\mathbb{R}^n)^2 = (-1)^{i(n-i)}\text{id}$ on $\mathcal{E}^i(\mathbb{R}^n)$ and $(*_\mathbb{R}^{n-1})^2 = (-1)^{(n-i)(i-1)}\text{id}$ on $\mathcal{E}^{n-i}(\mathbb{R}^{n-1})$, the last formula yields

$$*_\mathbb{R}^{n-1} \circ \mathcal{D}_{-a-1, a+1}^{n-i \rightarrow n-i} \circ (*_\mathbb{R}^n)^{-1} = (-1)^{i+1} \mathcal{D}_{n-2i-a-1, a+1}^{i \rightarrow i-1}.$$

Applying $*_{\mathbb{R}^{n-1}}$ from the left and $(*_\mathbb{R}^n)^{-1}$ from the right to the identity (13.7), we get

$$(-1)^{i+1} d_{\mathbb{R}^{n-1}}^* \mathcal{D}_{n-2i-a-1, a}^{i \rightarrow i} = -(-1)^{i+1} \gamma(-a-i + \frac{n-1}{2}, a) \mathcal{D}_{n-2i-a-1, a+1}^{i \rightarrow i-1}.$$

Thus Theorem 13.4 (3) is proved. \square

Proof of Theorem 13.4 (4). By the formula (1.5) of $\mathcal{D}_{u,a}^{i \rightarrow i-1}$, we have

$$d_{\mathbb{R}^{n-1}}^* \circ \mathcal{D}_{n-2i-a, a}^{i \rightarrow i-1} = -d_{\mathbb{R}^{n-1}}^* \circ \text{Rest}_{x_n=0} \circ \mathcal{D}_{a-2}^{\mu+1} d_{\mathbb{R}^n}^* \iota_{\frac{\partial}{\partial x_n}} d_{\mathbb{R}^n}$$

because $(d_{\mathbb{R}^{n-1}}^*)^2 = 0$. By Lemma 8.14 (2) and Lemma 8.15 (2), the right-hand side vanishes because $(d_{\mathbb{R}^n}^*)^2 = 0$ and $(\iota_{\frac{\partial}{\partial x_n}})^2 = 0$. \square

Alternatively, Theorem 13.4 (3) and (4) can be deduced from Theorem 13.4 (1) and (2), respectively, by using the Hodge star operators.

13.8. Renormalized factorization identities. In Section 13.1, we have shown various factorization identities for (unnormalized) symmetry breaking operators. Now observe from Proposition 1.4 that $\mathcal{D}_{u,a}^{i \rightarrow i-1}$ and $\mathcal{D}_{u,a}^{i \rightarrow i}$ may vanish for specific parameters (u, a, i) . In this section, we discuss factorization identities for renormalized symmetry breaking operators $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i-1}$ and $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i}$ defined in (1.9) and (1.10), respectively. We do not pursue finding all the constants explicitly for factorization identities in this section, as they are directly computed from the theorems for the unnormalized case in Section 13.1 and from (13.8) and (13.9) below. Instead we shall formulate results for renormalized operators in a way that we can benefit the following two advantages:

- to find some further nontrivial factorization identities that were stated as $T_Y \circ 0 = 0$ or $0 \circ T_X = 0$ for unnormalized operators with specific parameters;
- to determine exactly when the composition of two nonzero intertwining operators vanish.

The latter view point plays an important role in the branching problem for subquotients of principal series representations, see [22]. For this purpose, we use the notations $I(i, \lambda)_\alpha$ and $J(j, \nu)_\beta$ for principal series representations of $G = O(n+1, 1)$ and $G' = O(n, 1)$, respectively, instead of $\mathcal{E}^i(S^n)_{u,\delta}$ and $\mathcal{E}^j(S^{n-1})_{v,\varepsilon}$ for the notation in conformal geometry.

By definitions (1.9) and (1.6) on one hand, and (1.10) and (1.4) on the other hand, we have

$$(13.8) \quad \mathcal{D}_{u,a}^{i \rightarrow i} = \begin{cases} \frac{1}{2}(a+u)\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i} & \text{if } i = 0 \text{ or } a = 0, \\ \tilde{\mathcal{D}}_{u,a}^{i \rightarrow i} & \text{if } i \neq 0 \text{ and } a \neq 0, \end{cases}$$

$$(13.9) \quad \mathcal{D}_{u,a}^{i \rightarrow i-1} = \begin{cases} \frac{1}{2}(a+u+2i-n)\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i-1} & \text{if } i = n \text{ or } a = 0, \\ \tilde{\mathcal{D}}_{u,a}^{i \rightarrow i-1} & \text{if } i \neq n \text{ and } a \neq 0. \end{cases}$$

Consider the following diagrams for $j = i$ and $j = i - 1$:

$$\begin{array}{ccc} I\left(i, \frac{n}{2} - \ell\right)_\alpha & \xrightarrow{\tilde{\mathcal{D}}_{\frac{n}{2}-i-\ell, a+2\ell}^{i \rightarrow j}} & J\left(j, \frac{n}{2} + a + \ell\right)_\beta, \\ \mathcal{T}_{2\ell}^{(i)} \downarrow & & \\ I\left(i, \frac{n}{2} + \ell\right)_\alpha & \xrightarrow{\tilde{\mathcal{D}}_{\frac{n}{2}-i+\ell, a}^{i \rightarrow j}} & \end{array} \quad \begin{array}{ccc} I\left(i, \frac{n-1}{2} - a - \ell\right)_\alpha & \xrightarrow{\tilde{\mathcal{D}}_{\frac{n-1}{2}-i-\ell-a, a}^{i \rightarrow j}} & J\left(j, \frac{n-1}{2} - \ell\right)_\beta \\ & \searrow \tilde{\mathcal{D}}_{\frac{n-1}{2}-i-\ell-a, a+2\ell}^{i \rightarrow j} & \downarrow \mathcal{T}_{2\ell}^{(j)} \\ & & J\left(j, \frac{n-1}{2} + \ell\right)_\beta, \end{array}$$

where parameters δ and ε are chosen according to Theorem 2.8 (iii). In what follows, we put

$$\begin{aligned} p_\pm &\equiv p_\pm(i, \ell, a) := \begin{cases} i \pm \ell - \frac{n}{2} & \text{if } a \neq 0, \\ \pm 2 & \text{if } a = 0, \end{cases} \\ q &\equiv q(i, \ell, a) := \begin{cases} i + \ell - \frac{n-1}{2} & \text{if } i \neq 0, a \neq 0, \\ -2 & \text{if } i \neq 0, a = 0, \\ -(\ell + \frac{n-1}{2}) & \text{if } i = 0, \end{cases} \\ r &\equiv r(i, \ell, a) := \begin{cases} i - \ell - \frac{n+1}{2} & \text{if } i \neq n, a \neq 0, \\ 2 & \text{if } i \neq n, a = 0, \\ -(\ell + \frac{n+1}{2}) & \text{if } i = n, \end{cases} \end{aligned}$$

and recall $K_{\ell,a} = \prod_{k=1}^{\ell} \left(\left[\frac{a}{2}\right] + k\right)$. Then the factorization identities for renormalized differential symmetry breaking operators $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow j}$ for $j \in \{i-1, i\}$ and Branson's conformally covariant operators $\mathcal{T}_{2\ell}^{(i)}$ or $\mathcal{T}_{2\ell}^{(j)}$ are given as follows.

Theorem 13.15. *Suppose $0 \leq i \leq n-1$, $a \in \mathbb{N}$ and $\ell \in \mathbb{N}_+$. Then*

- (1) $\tilde{\mathcal{D}}_{\frac{n}{2}-i+\ell, a}^{i \rightarrow i} \circ \mathcal{T}_{2\ell}^{(i)} = p_- K_{\ell,a} \tilde{\mathcal{D}}_{\frac{n}{2}-i-\ell, a+2\ell}^{i \rightarrow i}$.
- (2) $\mathcal{T}_{2\ell}^{(i)} \circ \tilde{\mathcal{D}}_{\frac{n-1}{2}-i-\ell-a, a}^{i \rightarrow i} = q K_{\ell,a} \tilde{\mathcal{D}}_{\frac{n-1}{2}-i-\ell-a, a+2\ell}^{i \rightarrow i}$.

Theorem 13.16. *Suppose $1 \leq i \leq n$, $a \in \mathbb{N}$ and $\ell \in \mathbb{N}_+$. Then*

$$\begin{aligned} (1) \quad & \tilde{\mathcal{D}}_{\frac{n}{2}-i+\ell,a}^{i \rightarrow i-1} \circ \mathcal{T}_{2\ell}^{(i)} = p_+ K_{\ell,a} \tilde{\mathcal{D}}_{\frac{n}{2}-i-\ell,a+2\ell}^{i \rightarrow i-1}. \\ (2) \quad & \mathcal{T}_{2\ell}'^{(i-1)} \circ \tilde{\mathcal{D}}_{\frac{n-1}{2}-i-\ell-a,a}^{i \rightarrow i-1} = r K_{\ell,a} \tilde{\mathcal{D}}_{\frac{n-1}{2}-i-\ell-a,a+2\ell}^{i \rightarrow i-1}. \end{aligned}$$

Since $K_{\ell,a} > 0$, by examining the condition on (i, ℓ, a) for the constants p_{\pm}, q, r vanish, we see that Theorems 13.15 and 13.16 determine exactly when the composition of the two nonzero operators vanish:

Corollary 13.17. *Suppose we are in the setting of Theorems 13.15 or 13.16.*

(1) $\tilde{\mathcal{D}}_{\frac{n}{2}-i+\ell,a}^{i \rightarrow i} \circ \mathcal{T}_{2\ell}^{(i)} \neq 0$ except for the following case:

$$\tilde{\mathcal{D}}_{0,a}^{i \rightarrow i} \circ \mathcal{T}_{2i-n}^{(i)} = 0 \quad \text{for } n \text{ even, } \frac{n+2}{2} \leq i \leq n-1, \text{ and } a \in \mathbb{N}_+.$$

(2) $\tilde{\mathcal{D}}_{\frac{n}{2}-i+\ell,a}^{i \rightarrow i-1} \circ \mathcal{T}_{2\ell}^{(i)} \neq 0$ except for the following case:

$$\tilde{\mathcal{D}}_{n-2i,a}^{i \rightarrow i-1} \circ \mathcal{T}_{n-2i}^{(i)} = 0 \quad \text{for } n \text{ even, } 1 \leq i \leq \frac{n-2}{2}, \text{ and } a \in \mathbb{N}_+.$$

(3) $\mathcal{T}_{2\ell}'^{(i-1)} \circ \tilde{\mathcal{D}}_{\frac{n-1}{2}-i-\ell-a,a}^{i \rightarrow i-1} \neq 0$ except for the following case:

$$\mathcal{T}_{2i-n-1}'^{(i-1)} \circ \tilde{\mathcal{D}}_{n-2i-a,a}^{i \rightarrow i-1} = 0 \quad \text{for } n \text{ odd, } \frac{n+3}{2} \leq i \leq n-1, \text{ and } a \in \mathbb{N}_+.$$

(4) $\mathcal{T}_{2\ell}'^{(i)} \circ \tilde{\mathcal{D}}_{\frac{n-1}{2}-i-\ell-a,a}^{i \rightarrow i} \neq 0$ except for the following case:

$$\mathcal{T}_{n-2i-1}'^{(i)} \circ \tilde{\mathcal{D}}_{-a,a}^{i \rightarrow i} = 0 \quad \text{for } n \text{ odd, } 1 \leq i \leq \frac{n-3}{2}, \text{ and } a \in \mathbb{N}_+.$$

Next we consider the factorization identities corresponding to Theorems 13.3 and 13.4. By focusing on the “two advantages” as we mentioned at the beginning of this section, we omit giving the explicit constants and formulate the results as follows:

Theorem 13.18. *Let $n \geq 2$.*

(1) *Let $a \in \mathbb{N}$ and $0 \leq i \leq n-1$. For*

$$I(i, i)_i \xrightarrow{d} I(i+1, i+1)_{i+1} \xrightarrow{\tilde{\mathcal{D}}_{0,a}^{i+1 \rightarrow i}} J(i, i+a+1)_i,$$

the following two conditions on (i, a, n) are equivalent:

- (i) $\tilde{\mathcal{D}}_{0,a}^{i+1 \rightarrow i} \circ d = 0$;
- (ii) n is even, $0 \leq i < \frac{n-2}{2}$, and $a = n - 2i - 2$.

We note that, for n even,

$$\widetilde{\mathcal{D}}_{0,0}^{\frac{n}{2} \rightarrow \frac{n-2}{2}} \circ d = \widetilde{\mathcal{D}}_{0,1}^{\frac{n-2}{2} \rightarrow \frac{n-2}{2}}.$$

(2) Let $a \in \mathbb{N}$ and $0 \leq i \leq n-2$. For

$$I(i, i)_i \xrightarrow{d} I(i+1, i+1)_{i+1} \xrightarrow{\widetilde{\mathcal{D}}_{0,a}^{i+1 \rightarrow i+1}} J(i+1, i+a+1)_{i+1},$$

the following two conditions on (i, a, n) are equivalent:

(i) $\widetilde{\mathcal{D}}_{0,a}^{i+1 \rightarrow i+1} \circ d = 0;$

(ii) $a \in \mathbb{N}_+.$

We note that

$$\widetilde{\mathcal{D}}_{0,0}^{i+1 \rightarrow i+1} \circ d = \widetilde{\mathcal{D}}_{0,1}^{i \rightarrow i+1} \quad \text{for } 0 \leq i \leq n-2.$$

(3) Let $a \in \mathbb{N}$ and $1 \leq i \leq n$. For

$$I(i, n-i)_i \xrightarrow{d^*} I(i-1, n-i+1)_{i-1} \xrightarrow{\widetilde{\mathcal{D}}_{n-2i+2,a}^{i-1 \rightarrow i-1}} J(i-1, n+i+a+1)_{i-1},$$

the following two conditions on (i, a, n) are equivalent:

(i) $\widetilde{\mathcal{D}}_{n-2i+2,a}^{i-1 \rightarrow i-1} \circ d^* = 0;$

(ii) n is even, $\frac{n+2}{2} < i \leq n$, and $a = 2i - n - 2$.

We note that, for n even,

$$\widetilde{\mathcal{D}}_{0,0}^{\frac{n}{2} \rightarrow \frac{n}{2}} \circ d^* = -\widetilde{\mathcal{D}}_{-2,1}^{\frac{n+2}{2} \rightarrow \frac{n}{2}}.$$

(4) Let $a \in \mathbb{N}$ and $2 \leq i \leq n$. For

$$I(i, n-i)_i \xrightarrow{d^*} I(i-1, n-i+1)_{i-1} \xrightarrow{\widetilde{\mathcal{D}}_{n-2i+2,a}^{i-1 \rightarrow i-2}} J(i-2, n-i+a+1)_{i-2},$$

the following two conditions on (i, a, n) are equivalent:

(i) $\widetilde{\mathcal{D}}_{n-2i+2,a}^{i-1 \rightarrow i-2} \circ d^* = 0;$

(ii) $a \in \mathbb{N}_+.$

We note that

$$\widetilde{\mathcal{D}}_{n-2i+2,0}^{i-1 \rightarrow i-2} \circ d^* = \widetilde{\mathcal{D}}_{n-2i,1}^{i \rightarrow i-2} \quad \text{for } 2 \leq i \leq n.$$

(5) Let $a \in \mathbb{N}$ and $1 \leq i \leq n-1$. For

$$I(i, i-a-1)_i \xrightarrow{\widetilde{\mathcal{D}}_{-a-1,a}^{i \rightarrow i-1}} J(i-1, i-1)_{i-1} \xrightarrow{d} J(i, i)_i,$$

the following two conditions on (i, a, n) are equivalent:

(i) $d \circ \widetilde{\mathcal{D}}_{-a-1,a}^{i \rightarrow i-1} = 0;$

(ii) n is odd, $\frac{n+1}{2} < i \leq n-1$, $a = 2i - n - 1$.

We note that, for n odd,

$$d \circ \tilde{\mathcal{D}}_{-1,0}^{\frac{n+1}{2} \rightarrow \frac{n-1}{2}} = -\tilde{\mathcal{D}}_{-1,1}^{\frac{n+1}{2} \rightarrow \frac{n+1}{2}}.$$

(6) Let $a \in \mathbb{N}$ and $0 \leq i \leq n-2$. For

$$I(i, i-a)_i \xrightarrow{\tilde{\mathcal{D}}_{-a,a}^{i \rightarrow i}} J(i, i)_i \xrightarrow{d} J(i+1, i+1)_{i+1},$$

the following two conditions on (i, a, n) are equivalent:

- (a) $d \circ \tilde{\mathcal{D}}_{-a,a}^{i \rightarrow i} = 0$;
- (b) $a \in \mathbb{N}_+$ and $1 \leq i \leq n-2$.

We note that

$$d \circ \tilde{\mathcal{D}}_{-a,a}^{i \rightarrow i} = \tilde{\mathcal{D}}_{-a,a+1}^{i \rightarrow i+1} \quad \text{if } a = 0 \text{ or } i = 0.$$

(7) Let $a \in \mathbb{N}$ and $0 \leq i \leq n-1$. For

$$I(i, n-i-a-1)_i \xrightarrow{\tilde{\mathcal{D}}_{n-2i-a-1,a}^{i \rightarrow i}} I(i, n-i-1)_i \xrightarrow{d^*} J(i-1, n-i)_{i-1},$$

the following two conditions on (i, a, n) are equivalent:

- (i) $d^* \circ \tilde{\mathcal{D}}_{n-2i-a-1,a}^{i \rightarrow i} = 0$
- (ii) n is odd, $0 \leq i < \frac{n-1}{2}$, and $a = n-2i-1$.

We note that, for n odd,

$$d^* \circ \tilde{\mathcal{D}}_{0,0}^{\frac{n-1}{2} \rightarrow \frac{n-1}{2}} = -\tilde{\mathcal{D}}_{0,1}^{\frac{n-1}{2} \rightarrow \frac{n-3}{2}}.$$

(8) Let $a \in \mathbb{N}$ and $2 \leq i \leq n$. For

$$I(i, n-i-a)_i \xrightarrow{\tilde{\mathcal{D}}_{n-2i-a,a}^{i \rightarrow i-1}} J(i-1, n-i)_{i-1} \xrightarrow{d^*} J(i-2, n-i+1)_{i-2},$$

the following two conditions on (i, a, n) are equivalent:

- (a) $d^* \circ \tilde{\mathcal{D}}_{n-2i-a,a}^{i \rightarrow i-1} = 0$;
- (b) $a \in \mathbb{N}_+$ and $2 \leq i \leq n-1$.

We note that

$$d^* \circ \tilde{\mathcal{D}}_{n-2i-a,a}^{i \rightarrow i-1} = -\tilde{\mathcal{D}}_{n-2i-a,a+1}^{i \rightarrow i-2} \quad \text{if } a = 0 \text{ or } i = n.$$

Proof. Each equivalence in (1)-(8) is shown by the corresponding factorization identities for unnormalized operators given in Theorems 13.3 or 13.4, and by (13.8) and (13.9).

The factorization identities for specific parameters can be verified directly from the definition (1.9)-(1.12) of the renormalized operators $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow j}$. In fact, these operators for the specific values in Theorem 13.18 are “degenerate” and of much simple forms. \square

14. APPENDIX: GEGENBAUER POLYNOMIALS

This chapter collects some properties of the Gegenbauer polynomials that we use throughout the paper, in particular, in the proof of the explicit formulæ for symmetry breaking differential operators (Theorems 1.5, 1.6, 1.7, and 1.8) and the factorization identities for special parameters (Theorems 13.1, 13.2, and 13.3). In Section 14.5, we give a proof of Theorem 6.7 that determines solutions to the F-system for symmetry breaking operators from $I(i, \lambda)_\alpha$ to $J(i-1, \nu)_\beta$ ($2 \leq i \leq n$).

14.1. Normalized Gegenbauer polynomials. For $\lambda \in \mathbb{C}$ and $\ell \in \mathbb{N}$, the Gegenbauer (or ultraspherical) polynomial $C_\ell^\lambda(z)$ is given by the following formula ([1, 6.4], [8, 3.15 (2)]):

$$\begin{aligned} C_\ell^\lambda(z) &:= \sum_{k=0}^{\lfloor \frac{\ell}{2} \rfloor} (-1)^k \frac{\Gamma(\ell - k + \lambda)}{\Gamma(\lambda) k! (\ell - 2k)!} (2z)^{\ell - 2k} \\ &= \frac{\Gamma(\ell + 2\lambda)}{\Gamma(2\lambda) \Gamma(\ell + 1)} {}_2F_1 \left(-\ell, \ell + 2\lambda; \lambda + \frac{1}{2}; \frac{1-z}{2} \right). \end{aligned}$$

The generating function for $C_\ell^\lambda(z)$ is

$$(14.1) \quad (1 - 2zr + r^2)^{-\lambda} = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} \frac{(2zr)^k}{(1 + r^2)^{k+\lambda}} = \sum_{\ell=0}^{\infty} C_\ell^\lambda(z) r^\ell,$$

and $C_\ell^\lambda(z)$ solves the Gegenbauer differential equation

$$G_\ell^\lambda f(z) = 0,$$

where G_ℓ^λ is the Gegenbauer differential operator given by

$$(14.2) \quad G_\ell^\lambda := (1 - z^2) \frac{d^2}{dz^2} - (2\lambda + 1)z \frac{d}{dz} + \ell(\ell + 2\lambda).$$

We note that $C_\ell^\lambda(z) \equiv 0$ if $\ell \geq 1$ and $\lambda = 0, -1, -2, \dots, -\lfloor \frac{\ell-1}{2} \rfloor$. As in [21], we renormalize the Gegenbauer polynomial by

$$(14.3) \quad \tilde{C}_\ell^\lambda(z) := \frac{\Gamma(\lambda)}{\Gamma(\lambda + \lfloor \frac{\ell+1}{2} \rfloor)} C_\ell^\lambda(z) = \frac{1}{\Gamma(\lambda + \lfloor \frac{\ell+1}{2} \rfloor)} \sum_{k=0}^{\lfloor \frac{\ell}{2} \rfloor} (-1)^k \frac{\Gamma(\ell - k + \lambda)}{k! (\ell - 2k)!} (2z)^{\ell - 2k}.$$

Then $\tilde{C}_\ell^\lambda(z)$ is a nonzero polynomial of degree ℓ for all $\lambda \in \mathbb{C}$ and $\ell \in \mathbb{N}$. Here are the first five renormalized Gegenbauer polynomials.

- $\tilde{C}_0^\lambda(z) = 1.$
- $\tilde{C}_1^\lambda(z) = 2z.$
- $\tilde{C}_2^\lambda(z) = 2(\lambda + 1)z^2 - 1.$

- $\tilde{C}_3^\lambda(z) = \frac{4}{3}(\lambda + 2)z^3 - 2z$.
- $\tilde{C}_4^\lambda(z) = \frac{2}{3}(\lambda + 2)(\lambda + 3)z^4 - 2(\lambda + 2)z^2 + \frac{1}{2}$.

Then the ℓ -inflated polynomial (see (9.1)) of $\tilde{C}_\ell^\lambda(z)$ is given by

$$(14.4) \quad \begin{aligned} (I_\ell \tilde{C}_\ell^\lambda)(x, y) &= x^{\frac{\ell}{2}} \tilde{C}_\ell^\lambda\left(\frac{y}{\sqrt{x}}\right) \\ &= \sum_{k=0}^{\lfloor \frac{\ell}{2} \rfloor} (-1)^k \frac{\Gamma(\ell - k + \lambda)}{\Gamma(\lambda + \lfloor \frac{\ell+1}{2} \rfloor) \Gamma(k+1) \Gamma(\ell - 2k + 1)} (2y)^{\ell-2k} x^k. \end{aligned}$$

For instance, $(I_0 \tilde{C}_0^\lambda)(x, y) = 1$, $(I_1 \tilde{C}_1^\lambda)(x, y) = 2y$, $(I_2 \tilde{C}_2^\lambda)(x, y) = 2(\lambda + 1)y^2 - x$, etc.

From (14.3), the coefficient of z^ℓ in $\tilde{C}_\ell^\lambda(z)$ is found to be

$$(14.5) \quad \frac{\Gamma(\lambda + \ell) 2^\ell}{\Gamma(\lambda + \lfloor \frac{\ell+1}{2} \rfloor) \ell!}.$$

The dimension of the space of polynomial solutions to the Gegenbauer differential equation $G_\ell^\lambda f(z) = 0$ is generically one, however, it jumps to two when $\lambda - \frac{1}{2} \in \mathbb{Z}$ and $1 - 2\ell \leq 2\lambda \leq -\ell$, for which we have found an interpretation in the representation theory of $SL(2, \mathbb{R})$ [21]. The renormalized Gegenbauer polynomial $\tilde{C}_\ell^\lambda(z)$ is characterized among polynomial solutions by the following ([21, Thm. 11.4]):

Fact 14.1. *For all $\lambda \in \mathbb{C}$ and $\ell \in \mathbb{N}$,*

$$\{f(z) \in \text{Pol}_\ell[z]_{\text{even}} : G_\ell^\lambda f(z) = 0\} = \mathbb{C} \tilde{C}_\ell^\lambda(z).$$

See (4.5) for the definition of $\text{Pol}_\ell[z]_{\text{even}}$.

In (4.7), we introduced the *imaginary* Gegenbauer differential operator

$$R_\ell^\lambda = -\frac{1}{2} \left((1+t^2) \frac{d^2}{dt^2} + (1+2\lambda)t \frac{d}{dt} - \ell(\ell+2\lambda) \right),$$

which is related with the Gegenbauer differential operator G_ℓ^ν defined in (14.2) as follows:

Lemma 14.2. *Let $f(z)$ be a polynomial in z , and $g(t) = f(z)$ with $z := e^{\frac{\pi\sqrt{-1}}{2}}t$. Then*

$$(14.6) \quad 2(R_\ell^\lambda g)(t) = (G_\ell^\lambda f)(z).$$

Proof. Direct from $\frac{d}{dt} = e^{\frac{\pi\sqrt{-1}}{2}} \frac{d}{dz}$. □

Therefore, Fact 14.1 implies the following:

Lemma 14.3. *For any $\lambda \in \mathbb{C}$ and $\ell \in \mathbb{N}$, the $\text{Pol}_\ell[t]_{\text{even}}$ -solution space of the ordinary differential equation $R_\ell^\lambda g(t) = 0$ is one-dimensional. Moreover, it is spanned by $\tilde{C}_\ell^\lambda \left(e^{\frac{\pi\sqrt{-1}}{2}} t \right)$.*

14.2. Derivatives of Gegenbauer polynomials. For $\mu \in \mathbb{C}$ and $\ell \in \mathbb{N}$, we recall from (1.3)

$$\gamma(\mu, \ell) = \frac{\Gamma(\mu + 1 + [\frac{\ell}{2}])}{\Gamma(\mu + [\frac{\ell+1}{2}])} = \begin{cases} 1 & \text{if } \ell \text{ is odd,} \\ \mu + \frac{\ell}{2} & \text{if } \ell \text{ is even.} \end{cases}$$

We collect two formulæ about the first derivative of the renormalized Gegenbauer polynomial $\tilde{C}_\ell^\mu(z)$.

Lemma 14.4. *Let $\mu \in \mathbb{C}$ and $\ell \in \mathbb{N}$.*

$$(14.7) \quad \frac{d}{dz} \tilde{C}_\ell^\mu(z) = 2\gamma(\mu, \ell) \tilde{C}_{\ell-1}^{\mu+1}(z),$$

$$(14.8) \quad \left(z \frac{d}{dz} - \ell \right) \tilde{C}_\ell^\mu(z) = 2\tilde{C}_{\ell-2}^{\mu+1}(z).$$

Proof. The first identity (14.7) follows from $\frac{d}{dz} C_\ell^\mu(z) = 2\mu C_{\ell-1}^{\mu+1}(z)$ (see [1, (6.4.15)], [8, 3.15.2 (30)] for example). To see the second identity, let $\vartheta_z := z \frac{\partial}{\partial z}$ and $\vartheta_r = r \frac{\partial}{\partial r}$. Applying $\vartheta_z - \vartheta_r$ to (14.1), we get

$$2\lambda r^2 \sum_{\ell=0}^{\infty} C_\ell^{\lambda+1}(z) r^\ell = \sum_{\ell=0}^{\infty} (\vartheta_z - \ell) C_\ell^\lambda(z) r^\ell,$$

whence $(\vartheta_z - \ell) C_\ell^\lambda(z) = 2\lambda C_{\ell-2}^{\lambda+1}(z)$. By (14.3), we get $(\vartheta_z - \ell) \tilde{C}_\ell^\lambda(z) = 2\tilde{C}_{\ell-2}^{\lambda+1}(z)$. \square

14.3. Three-term relations among renormalized Gegenbauer polynomials.

In this section we collect three-term relations for renormalized Gegenbauer polynomials \tilde{C}_ℓ^μ for $\mu \in \mathbb{C}$. Further identities for special values μ will be treated in the next section.

We begin with useful identities for Gegenbauer differential operators G_ℓ^μ (see (14.2)):

Lemma 14.5. *Let $\mu \in \mathbb{C}$ and $\ell \in \mathbb{N}$.*

$$(14.9) \quad G_\ell^{\mu+1} - G_\ell^\mu = -2 \left(z \frac{d}{dz} - \ell \right).$$

$$(14.10) \quad G_\ell^{\mu+1} - G_{\ell-2}^{\mu+1} = 4(\mu + \ell).$$

$$(14.11) \quad G_\ell^\mu z - z G_{\ell-1}^{\mu+1} = 2 \frac{d}{dz}.$$

$$(14.12) \quad G_\ell^{\mu-1} - G_{\ell-2}^\mu = 2(\vartheta_z + \ell + 2\mu - 2).$$

$$(14.13) \quad G_\ell^{\mu-1}(z^2 - 1) - (z^2 - 1)G_{\ell-2}^{\mu+1} = -2(2\mu - 1).$$

$$(14.14) \quad (z^2 - 1)^\ell G_\ell^{\frac{1}{2}+\ell} (z^2 - 1)^{-\ell} = G_{\ell+2\ell}^{\frac{1}{2}-\ell}.$$

Proof. The first four formulæ are easily obtained by the definition (14.2). For instance, the third one is obtained by the following commutation relations

$$\frac{d}{dz}z - z \frac{d}{dz} = 1, \quad \frac{d^2}{dz^2}z - z \frac{d^2}{dz^2} = 2 \frac{d}{dz}$$

in the Weyl algebra $\mathcal{D}(\mathbb{C}) = \mathbb{C}[z, \frac{d}{dz}]$. To see the sixth one, we apply the following identities:

$$(z^2 - 1)^{\ell+1} \frac{d}{dz} (z^2 - 1)^{-\ell} = (z^2 - 1) \frac{d}{dz} - 2\ell z,$$

$$(z^2 - 1)^{\ell+2} \frac{d^2}{dz^2} (z^2 - 1)^{-\ell} = (z^2 - 1)^2 \frac{d^2}{dz^2} - 4\ell z (z^2 - 1) \frac{d}{dz} + 2\ell((2\ell + 1)z^2 + 1).$$

Now (14.14) follows from the definition (14.2) of the operator G_ℓ^λ . \square

Lemma 14.6. *Let $\ell \in \mathbb{N}$ and $\mu \in \mathbb{C}$. Then*

$$(14.15) \quad (\mu + \ell) \tilde{C}_\ell^\mu(z) + \tilde{C}_{\ell-2}^{\mu+1}(z) = \left(\mu + \left\lceil \frac{\ell+1}{2} \right\rceil \right) \tilde{C}_\ell^{\mu+1}(z).$$

Proof. By the relations (14.9) and (14.10) for Gegenbauer differential operators, we have

$$G_\ell^{\mu+1}((\mu + \ell) \tilde{C}_\ell^\mu(z) + \tilde{C}_{\ell-2}^{\mu+1}(z)) = -2(\mu + \ell) \left(\left(z \frac{d}{dz} - \ell \right) \tilde{C}_\ell^\mu(z) - 2 \tilde{C}_{\ell-2}^{\mu+1}(z) \right) = 0.$$

The second equality follows from (14.8). Since $(\mu + \ell) \tilde{C}_\ell^\mu(z) + \tilde{C}_{\ell-2}^{\mu+1}(z) \in \text{Pol}_\ell[z]_{\text{even}}$, according to Fact 14.1 there exists $A \in \mathbb{C}$ such that

$$(\mu + \ell) \tilde{C}_\ell^\mu(z) + \tilde{C}_{\ell-2}^{\mu+1}(z) = A \tilde{C}_\ell^{\mu+1}(z).$$

Comparing the coefficients of the leading term z^a in the both sides by using (14.5), we get $A = \mu + \left\lceil \frac{a+1}{2} \right\rceil$. \square

Lemma 14.7. *Let $\ell \in \mathbb{N}$ and $\mu \in \mathbb{C}$. Then we have*

$$(14.16) \quad \gamma(\mu, \ell)(z^2 - 1)\tilde{C}_{\ell-1}^{\mu+1}(z) + (\mu - \frac{1}{2})z\tilde{C}_{\ell}^{\mu}(z) = \frac{1}{2}(\ell + 1)\gamma(\mu - \frac{1}{2}, \ell)\tilde{C}_{\ell+1}^{\mu-1}(z).$$

Proof. We apply (14.7) to the left-hand side of the following formula (see [8, 3.15 (10)]):

$$\frac{d}{dz} \left((1 - z^2)^{\mu-\frac{1}{2}} C_{\ell}^{\mu}(z) \right) = \frac{(\ell + 1)(\ell + 2\mu - 1)}{2(1 - \mu)} (1 - z^2)^{\mu-\frac{3}{2}} C_{\ell+1}^{\mu-1}(z).$$

By using the identity $2\gamma(\mu, \ell - 1)\gamma(\mu - \frac{1}{2}, \ell) = \ell + 2\mu - 1$, we see that the renormalization (14.3) gives the formula (14.16). \square

Lemma 14.8. *Let $\ell \in \mathbb{N}$ and $\mu \in \mathbb{C}$. Then*

$$(14.17) \quad (z^2 - 1)\tilde{C}_{\ell-2}^{\mu+1}(z) + (\mu - \frac{1}{2})\tilde{C}_{\ell}^{\mu}(z) = (\mu + \left\lfloor \frac{\ell}{2} \right\rfloor - \frac{1}{2})\tilde{C}_{\ell}^{\mu-1}(z).$$

Proof. It follows from the identities (14.13) and (14.9) in the Weyl algebra that

$$\begin{aligned} & G_{\ell}^{\mu-1} \left((z^2 - 1)\tilde{C}_{\ell-2}^{\mu+1}(z) + (\mu - \frac{1}{2})\tilde{C}_{\ell}^{\mu}(z) \right) \\ &= (2\mu - 1) \left(\left(z \frac{d}{dz} - \ell \right) \tilde{C}_{\ell}^{\mu}(z) - 2\tilde{C}_{\ell-2}^{\mu+1}(z) \right), \end{aligned}$$

which vanishes by (14.8). By the uniqueness of the solutions to $G_{\ell}^{\mu-1}f(z) = 0$ for $f \in \text{Pol}_{\ell}[z]_{\text{even}}$ (see Fact 14.1), there exists $c \in \mathbb{C}$, such that

$$(z^2 - 1)\tilde{C}_{\ell-2}^{\mu+1}(z) + (\mu - \frac{1}{2})\tilde{C}_{\ell}^{\mu}(z) = c\tilde{C}_{\ell}^{\mu-1}(z).$$

Comparing the coefficients of the leading term z^{ℓ} by (14.5), and using the identity

$$(14.18) \quad 4(\mu + \left\lfloor \frac{\ell}{2} \right\rfloor - \frac{1}{2})(\mu + \left\lfloor \frac{\ell-1}{2} \right\rfloor) = \ell^2 - \ell + 2(2\mu - 1)(\mu + \ell - 1),$$

we conclude that $c = \mu + \left\lfloor \frac{\ell}{2} \right\rfloor - \frac{1}{2}$. \square

Lemma 14.9. *Let $\ell \in \mathbb{N}$ and $\mu \in \mathbb{C}$. Then we have*

$$(14.19) \quad \tilde{C}_{\ell-2}^{\mu+1}(z) = \gamma(\mu, \ell)z\tilde{C}_{\ell-1}^{\mu+1}(z) - \frac{\ell}{2}\tilde{C}_{\ell}^{\mu}(z).$$

Proof. The formula is a direct consequence of (14.7) and (14.8). Alternatively, the lemma is derived from the following three-term relation (see [8, 3.15 (27)]):

$$(14.20) \quad \ell C_{\ell}^{\mu}(z) = -2\mu (zC_{\ell-1}^{\mu+1}(z) - C_{\ell-2}^{\mu+1}(z)).$$

\square

Lemma 14.10. *For $\ell \in \mathbb{N}$ and $\mu \in \mathbb{C}$,*

$$(14.21) \quad z\tilde{C}_{\ell-1}^{\mu+1}(z) - \gamma(\mu, \ell+1)\tilde{C}_{\ell}^{\mu+1}(z) + \gamma(\mu - \frac{1}{2}, \ell+1)\tilde{C}_{\ell}^{\mu}(z) = 0.$$

Proof. By (14.10) and (14.11), we get

$$G_{\ell}^{\mu} \left(z\tilde{C}_{\ell-1}^{\mu+1}(z) - \gamma(\mu, \ell+1)\tilde{C}_{\ell}^{\mu+1}(z) \right) = 2 \frac{d}{dz} \tilde{C}_{\ell-1}^{\mu+1}(z) - 2\gamma(\mu, \ell+1) \left(z \frac{d}{dz} - \ell \right) \tilde{C}_{\ell}^{\mu+1}(z).$$

By (14.7) and (14.8), this amounts to

$$4\gamma(\mu+1, \ell-1)\tilde{C}_{\ell-2}^{\mu+2}(z) - 4\gamma(\mu, \ell+1)\tilde{C}_{\ell-2}^{\mu+2}(z) = 0,$$

because $\gamma(\mu+1, \ell-1) = \gamma(\mu, \ell+1)$.

Since $z\tilde{C}_{\ell-1}^{\mu+1}(z) - \gamma(\mu, \ell+1)\tilde{C}_{\ell}^{\mu+1}(z) \in \text{Pol}_{\ell}[z]_{\text{even}}$, there exists $A \in \mathbb{C}$ by Fact 14.1 such that

$$z\tilde{C}_{\ell-1}^{\mu+1}(z) - \gamma(\mu, \ell+1)\tilde{C}_{\ell}^{\mu+1}(z) = A\tilde{C}_{\ell}^{\mu}(z).$$

Comparing the coefficients of the leading terms z^{ℓ} on both sides, we have

$$A = -\frac{\gamma(\mu, \ell+1)(\ell+2\mu)}{2(\mu + \lceil \frac{\ell+1}{2} \rceil)} = -\gamma(\mu - \frac{1}{2}, \ell+1).$$

Alternatively, the lemma follows directly from the three-term relation [8, 3.15 (28)] for the corresponding (unnormalized) Gegenbauer polynomials. \square

14.4. Duality of Gegenbauer polynomials for special values. We recall from (1.13) that $K_{\ell,a} = \prod_{j=1}^{\ell} \left(\lceil \frac{a}{2} \rceil + j \right)$ is a positive integer for any $\ell, a \in \mathbb{N}$.

Proposition 14.11. *Let $a, \ell \in \mathbb{N}$. Then*

$$(14.22) \quad \tilde{C}_a^{-a-\ell}(z) = (-1)^{\ell} K_{\ell,a} \tilde{C}_{a+2\ell}^{-a-\ell}(z),$$

$$(14.23) \quad (z^2 - 1)^{\ell} \tilde{C}_a^{\frac{1}{2}+\ell}(z) = K_{\ell,a} \tilde{C}_{a+2\ell}^{\frac{1}{2}-\ell}(z).$$

Proof. The first equality (14.22) was proved in [19, Lem. 4.12]. We thus give a proof of the second equality (14.23). Since $G_a^{\frac{1}{2}+\ell} \tilde{C}_a^{\frac{1}{2}+\ell}(z) = 0$, we get from (14.14)

$$G_{a+2\ell}^{\frac{1}{2}-\ell} \left((z^2 - 1)^{\ell} \tilde{C}_a^{\frac{1}{2}+\ell}(z) \right) = 0.$$

Since $(z^2 - 1)^{\ell} \tilde{C}_a^{\frac{1}{2}+\ell}(z) \in \text{Pol}_{a+2\ell}[z]_{\text{even}}$, there exists $A \in \mathbb{C}$ such that

$$(z^2 - 1)^{\ell} \tilde{C}_a^{\frac{1}{2}+\ell}(z) = A\tilde{C}_{a+2\ell}^{\frac{1}{2}-\ell}(z)$$

by Fact 14.1. Comparing the coefficients of the leading term $z^{a+2\ell}$ by (14.5), we have

$$A = \frac{\Gamma(\frac{1}{2} + \lceil \frac{a+1}{2} \rceil) (a+2\ell)!}{2^{2\ell} \Gamma(\frac{1}{2} + \ell + \lceil \frac{a+1}{2} \rceil) a!} = \prod_{j=1}^{\ell} \left(\lceil \frac{a}{2} \rceil + j \right) = K_{\ell,a}.$$

Hence (14.23) is proved. \square

14.5. Proof of Theorem 6.7. As an application of the three-term relations of (renormalized) Gegenbauer polynomials developed in Section 14.3, we give a proof of Theorem 6.7 (solving the F-system) in this section.

Let $a, i \in \mathbb{N}$ and $\mu \in \mathbb{C}$. We define a linear isomorphism $\Psi \equiv \Psi(a, i, \mu, n)$ by

$$(14.24) \quad \Psi: \bigoplus_{k=0}^2 \text{Pol}_{a-k}[t]_{\text{even}} \xrightarrow{\sim} \bigoplus_{k=0}^2 \text{Pol}_{a-k}[z]_{\text{even}}, \quad (g_0, g_1, g_2) \mapsto (f_0, f_1, f_2)$$

with the following relations: $z = e^{\frac{\pi\sqrt{-1}}{2}}t$ and

$$\begin{aligned} g_2(t) &= f_2(z), \\ g_1(t) &= e^{-\frac{\pi\sqrt{-1}}{2}} f_1(z), \\ g_0(t) &= \begin{cases} f_0 & \text{if } a = 0, \\ f_0(z) - \frac{1}{a} \left(a + \mu - \frac{n+3}{2} + i \right) z f_1(z) & \text{if } a \in \mathbb{N}_+. \end{cases} \end{aligned}$$

Via the isomorphism (14.24), the convention (6.14) for $(g_0(t), g_1(t), g_2(t))$ is translated into the following one for $(f_0(z), f_1(z), f_2(z))$:

$$(14.25) \quad f_1 = f_2 = 0 \quad \text{for } a = 0; \quad f_2 = 0 \quad \text{for } a = 1; \quad f_2 = 0 \quad \text{for } i = 1; \quad f_1 = f_2 = 0 \quad \text{for } i = n.$$

In connection to the F-system for symmetry breaking operator from $I(i, \lambda)_\alpha$ to $J(i - 1, \nu)_\beta$, the parameter $\lambda \in \mathbb{C}$ in the principal series representation $I(i, \lambda)_\alpha$ will be related as

$$\mu = \lambda - \frac{n-3}{2}.$$

If $(g_0(t), g_1(t), g_2(t)) = ((6.2), (6.3), (6.4))$ in Theorem 6.1, then

$$(14.26) \quad \Psi(g_0, g_1, g_2) = \left(C\tilde{C}_{a-2}^\mu(z), A\tilde{C}_{a-1}^\mu(z), \tilde{C}_{a-2}^\mu(z) \right)$$

with

$$(14.27) \quad A := \gamma(\mu - 1, a), \quad C := \frac{\lambda - n + i}{a} + \frac{i - 1}{n - 1} = \frac{1}{a} \left(\mu - \frac{n+3}{2} + i \right) + \frac{i - 1}{n - 1}.$$

In what follows, we denote by (L_j) the differential equation $L_j(g_0, g_1, g_2) = 0$ (see (6.6)-(6.13)) for simplicity. Then, via the transformation Ψ , we observe that the differential equations (L_j) for (g_0, g_1, g_2) in Section 6.2 are transferred to differential equations for (f_0, f_1, f_2) as follows:

Lemma 14.12. *Via the isomorphism (14.24), the triple $(g_0(t), g_1(t), g_2(t))$ satisfies (L_j) if and only if $(f_0(z), f_1(z), f_2(z))$ satisfies the corresponding differential equation (G_j) for each $j = 1, 2, \dots, 9$, where we set:*

$$(G1) \quad G_{a-2}^\mu f_2(z) = 0,$$

$$(G2) \quad G_{a-1}^\mu f_1(z) = 0,$$

$$(G3) \quad (\vartheta_z - a + 1)f_1(z) - \frac{df_2}{dz}(z) = 0,$$

$$(G4) \quad (\vartheta_z + 2\mu + a - 2)f_2(z) - \frac{df_1}{dz}(z) = 0,$$

$$(G5) \quad \frac{df_0}{dz}(z) - \frac{1}{a} \left(a + \mu - \frac{n+3}{2} + i \right) (\vartheta_z - a + 1)f_1(z) + \frac{n-i}{n-1} \frac{df_2}{dz}(z) = 0,$$

$$(G6) \quad af_0(z) = \left(\mu - \frac{n+3}{2} + i + \frac{a(i-1)}{n-1} \right) f_2(z) \\ + e^{-\frac{\pi\sqrt{-1}}{2}} z \left(\frac{df_0}{dz}(z) - \frac{1}{a} \left(a + \mu - \frac{n+3}{2} + i \right) (\vartheta_z - a + 1)f_1(z) + \frac{n-i}{n-1} \frac{df_2}{dz}(z) \right),$$

$$(G7) \quad \frac{1}{2} G_a^{\mu-1} \left(f_0(z) - \frac{1}{a} \left(a + \mu - \frac{n+3}{2} + i \right) z f_1(z) \right) + \frac{n-i}{n-1} \frac{df_1}{dz}(z) = 0,$$

$$(G8) \quad af_0(z) = \left(\mu - \frac{n+3}{2} + i + \frac{a(i-1)}{n-1} \right) f_2(z),$$

$$(G9) \quad \frac{df_0}{dz}(z) - \left(\frac{i-1}{n-1} + \frac{1}{a} \left(\mu - \frac{n+3}{2} + i \right) \right) (\vartheta_z - a + 1)f_1(z) = 0.$$

With the constants A, C as in (14.27), we define polynomials $F_k(z)$ ($k = 0, 1, 2$) by

- (1) $i = 1, a \geq 1 :$ $(F_0(z), F_1(z), F_2(z)) := (C\tilde{C}_{a-2}^\mu(z), A\tilde{C}_{a-1}^\mu(z), 0);$
- (2) $2 \leq i \leq n-1, a \geq 1 :$ $(F_0(z), F_1(z), F_2(z)) := (C\tilde{C}_{a-2}^\mu(z), A\tilde{C}_{a-1}^\mu(z), \tilde{C}_{a-2}^\mu(z));$
- (3) $i = n, a \geq 1 :$ $(F_0(z), F_1(z), F_2(z)) := (\tilde{C}_{a-2}^\mu(z), 0, 0);$
- (4) $1 \leq i \leq n, a = 0 :$ $(F_0(z), F_1(z), F_2(z)) := (1, 0, 0).$

Note that $F_0 = C\tilde{C}_{a-2}^\mu(z) \in \text{Pol}_{a-2}[z]_{\text{even}} \subset \text{Pol}_a[z]_{\text{even}}$. Then, Theorem 6.7 is equivalent to the following assertion via the transformation Ψ .

Proposition 14.13. *Let $n \geq 3$ and $1 \leq i \leq n$. Suppose $f_k(z) \in \text{Pol}_{a-k}[z]_{\text{even}}$ ($k = 0, 1, 2$) with the convention (14.25). Then, up to scalar multiple, the solution*

(f_0, f_1, f_2) to

$$\begin{aligned} (G2), (G7), (G9) & \quad i = 1, \\ (Gr) \quad (r = 1, 2, \dots, 7) & \quad 2 \leq i \leq n-1, \\ (G1) & \quad i = n, \end{aligned}$$

is given by (F_0, F_1, F_2) .

Proposition 14.13 in the case $a = 0$ is trivial. Since $f_1 = f_2 = 0$ for $i = n$, Proposition 14.13 in the case $i = n$ is clear from Fact 14.1 because $(G7)$ is reduced to the Gegenbauer differential equation $G_a^{\mu-1} f_0 = 0$.

For $2 \leq i \leq n-1$, the proof of Proposition 14.13 is divided into Lemmas 14.14 and 14.15 below.

Lemma 14.14. *Suppose that $a \geq 1$ and $\mu \in \mathbb{C}$. Then we have*

$$\dim_{\mathbb{C}} \left\{ (f_0, f_1, f_2) \in \bigoplus_{k=0}^2 \text{Pol}_{a-k}[z]_{\text{even}} : (f_0, f_1, f_2) \text{ solves } (Gj), j = 1, 2, 3, 4, 8 \right\} \leq 1.$$

The following lemma shows that the left-hand side is equal to one.

Lemma 14.15. *Suppose $2 \leq i \leq n-1$. Then, for any $a \in \mathbb{N}_+$ and $\mu \in \mathbb{C}$, the triple (F_0, F_1, F_2) solves (Gj) for all $j = 1, \dots, 9$.*

Proof of Lemma 14.14. We shall prove that $(f_0, f_1, f_2) \in \bigoplus_{k=0}^2 \text{Pol}_{a-k}[z]_{\text{even}}$ satisfies (Gj) for $j = 1, 2, 3, 4, 8$ only if $(f_0, f_1, f_2) = p(F_0, F_1, F_2)$ for some $p \in \mathbb{C}$. We consider the cases $a = 1$ and $a \geq 2$, separately.

1) $a = 1$: If $a = 1$, then $f_2 = 0$ by (14.25). In turn, $f_0 = 0$ by $(G8)$. Since $\text{Pol}_{a-k}[t]_{\text{even}} = \mathbb{C} \cdot 1$ for $a = k = 1$, f_1 is a constant function. Thus $(f_0, f_1, f_2) \in \mathbb{C}(0, 1, 0) = \mathbb{C}(F_0, F_1, F_2)$.

2) $a \geq 2$: First we apply Fact 14.1 to see that the polynomial solutions to $(G1)$ $(G2)$ are of the form $f_2(z) = p\tilde{C}_{a-2}^{\mu}(z)(= pF_2(z))$, $f_1(z) = q\tilde{C}_{a-1}^{\mu}(z)$ for some $p, q \in \mathbb{C}$. It then follows from (14.7) and (14.8) that $(G3)$ is equivalent to

$$(G3)' \quad 2(q - p\gamma(\mu, a-2))\tilde{C}_{a-3}^{\mu+1}(z) = 0,$$

whence we get for $a \geq 3$

$$(14.28) \quad q = p\gamma(\mu, a-2) = p\gamma(\mu-1, a).$$

Similarly it follows from (14.7) and (14.8), and Lemma 14.6 that $(G4)$ is equivalent to

$$(G4)' \quad 2\gamma(\mu, a-1)(p\gamma(\mu, a-2) - q)\tilde{C}_{a-2}^{\mu+1}(z) = 0,$$

where we have used the identity

$$(14.29) \quad \gamma(\mu, a-1)\gamma(\mu, a-2) = \mu + \left\lfloor \frac{a-1}{2} \right\rfloor.$$

Hence (14.28) holds if $\gamma(\mu, a-1) \neq 0$, in particular, if $a = 2$. Thus $f_1 = q\tilde{C}_{a-1}^\mu = p\gamma(\mu-1, a)\tilde{C}_{a-1}^\mu = pF_1$ for any $a \geq 2$.

Finally, (G8) is equivalent to $f_0 = Cf_2$ with C in (14.27). Since $f_2 = pF_2$ and $F_0 = CF_2$, this implies $f_0(z) = CpF_2(z) = pF_0(z)$. Hence $(f_0, f_1, f_2) = p(F_0, F_1, F_2)$, and the proof is completed. \square

Proof of Lemma 14.15. We consider the cases that $a = 1$ and $a \geq 2$, separately.

1) $a = 1$: Obviously, $(F_0, F_1, F_2) = (0, 1, 0)$ satisfies the equations (G1)-(G9).

2) $a \geq 2$: (F_0, F_1, F_2) satisfies (Gr) ($r = 1, 2, 8$) by the definition of (F_0, F_1, F_2) and Fact 14.1, and (Gr) ($r = 3, 4$) as is in the proof of Lemma 14.14. Thus it remains to prove that (F_0, F_1, F_2) solves (G5) and (G7). (We recall that (G6) and (G9) are linear combinations of the others.)

For $(f_0, f_1, f_2) = (F_0, F_1, F_2)$, the equation (G5) amounts to

$$2 \left(C + \frac{n-i}{n-1} \right) \gamma(\mu, a-2)\tilde{C}_{a-3}^{\mu+1}(z) - \frac{2}{a} \left(a + \mu - \frac{n+3}{2} + i \right) A\tilde{C}_{a-3}^{\mu+1}(z) = 0$$

by (14.7) and (14.8). This identity obviously holds by the definition (14.27) of the constants A and C .

Finally, let us verify that the triple (F_0, F_1, F_2) satisfies the equation (G7). For this we use (14.11) and (14.12). Since $G_{a-2}^\mu F_0 = G_{a-1}^\mu F_1 = 0$, the left-hand side of (G7) applied to $(f_0, f_1, f_2) = (F_0, F_1, F_2)$ amounts to

$$\begin{aligned} & (\vartheta_z + a + 2\mu - 2)F_0(z) - \frac{1}{a} \left(a + \mu - \frac{n+3}{2} + i \right) \frac{dF_1}{dz}(z) + \frac{n-i}{n-1} \frac{dF_1}{dz}(z) \\ &= (\vartheta_z - a + 2)F_0(z) + 2(a + \mu - 2)F_0(z) - C \frac{dF_1}{dz}(z) \\ &= C \left((\vartheta_z - a + 2)\tilde{C}_{a-2}^\mu(z) + 2(a + \mu - 2)\tilde{C}_{a-2}^\mu(z) - \gamma(\mu-1, a) \frac{d}{dz} \tilde{C}_{a-1}^\mu(z) \right). \end{aligned}$$

By (14.7), (14.8) and (14.29) again, this equals

$$2C \left(\tilde{C}_{a-4}^{\mu+1}(z) + (a + \mu - 2)\tilde{C}_{a-2}^\mu(z) - \left(\mu + \left\lfloor \frac{a-1}{2} \right\rfloor \right) \tilde{C}_{a-2}^{\mu+1}(z) \right),$$

which vanishes by the three-term relation given in (14.15). Hence the proof of Lemma 14.15 is complete. \square

Thus Proposition 14.13 for $2 \leq i \leq n-1$ is proved.

Finally, let us consider Proposition 14.13 in the case $i = 1$. It is sufficient to show:

Lemma 14.16. *For any $a \in \mathbb{N}_+$ and $\mu \in \mathbb{C}$, we have*

$$\left\{ (f_0, f_1) \in \bigoplus_{k=0}^1 \text{Pol}_{a-k}[z]_{\text{even}} : (f_0, f_1, 0) \text{ solves } (G2), (G7), (G9) \right\} = \mathbb{C}(F_0, F_1).$$

Proof of Lemma 14.16. Since (F_0, F_1, F_2) solves (Gr) for all $r = 1, \dots, 9$, and since (Gr) ($r = 2, 7, 9$) does not involve g_2 , we conclude that $(F_0, F_1, 0)$ also solves (Gr) ($r = 2, 7, 9$).

Conversely, let us show $(f_0, f_1) \in \mathbb{C}(F_0, F_1)$ if $(f_0, f_1, 0)$ satisfies $(G2)$, $(G7)$, and $(G9)$. We observe that for $i = 1$, $(G7)$ and $(G9)$ amount to

$$\begin{aligned} (G7) \quad & \frac{1}{2} G_a^{\mu-1} \left(f_0(z) - \frac{1}{a} \left(a + \mu - \frac{n+1}{2} \right) z f_1(z) \right) + \frac{df_1}{dz}(z) = 0, \\ (G9) \quad & \frac{df_0}{dz}(z) - \frac{1}{a} \left(\mu - \frac{n+1}{2} \right) (\vartheta_z - a + 1) f_1(z) = 0, \end{aligned}$$

respectively. By $(G2)$, we have $f_1(z) = p \tilde{C}_{a-1}^\mu(z)$ for some constant p . It follows from (14.8) that $(\vartheta_z - a + 1) f_1(z) = 2p \tilde{C}_{a-3}^{\mu+1}(z)$. Thus $(G9)$ amounts to

$$(14.30) \quad \frac{df_0}{dz}(z) = \frac{2}{a} p \left(\mu - \frac{n+1}{2} \right) \tilde{C}_{a-3}^{\mu+1}(z).$$

By (14.7), $f_0(z)$ is then of the form $f_0(z) = q_1 \tilde{C}_{a-2}^\mu(z) + q_2$, where q_1 and q_2 are some constants satisfying

$$(14.31) \quad q_1 \gamma(\mu - 1, a) = \frac{1}{a} p \left(\mu - \frac{n+1}{2} \right).$$

Thus, for $f_1(z) = p \tilde{C}_{a-1}^\mu(z)$ and $f_0(z) = q_1 \tilde{C}_{a-2}^\mu(z) + q_2$ with (14.31), by using the identities (14.11) and (14.12) of the Gegenbauer differential operator G_ℓ^μ 's and the three-term relation (14.15), we see that $(G7)$ implies

$$0 = \frac{1}{2} G_a^{\mu-1} \left(f_0(z) - \frac{1}{a} \left(a + \mu - \frac{n+1}{2} \right) z f_1(z) \right) + \frac{df_1}{dz} = \frac{1}{2} G_a^{\mu-1} q_2.$$

Therefore $q_2 = 0$ and so $f_0(z) = q_1 \tilde{C}_{a-2}^\mu(z)$. It is then clear from (14.31) that (f_0, f_1) is proportional to (F_0, F_1) when $\gamma(\mu - 1, a) \neq 0$.

Now suppose that $\gamma(\mu - 1, a) = 0$. Since in this case we have $(F_0, F_1) \in \mathbb{C}(\tilde{C}_{a-2}^\mu, 0)$, it suffices to show $p = 0$ for $f_1(z) = p \tilde{C}_{a-1}^\mu(z)$. It follows from (1.3) that if $\gamma(\mu - 1, a) = 0$, then $\mu - 1 + \frac{a}{2} = 0$. If $\mu - \frac{n+1}{2} = 0$, then, as n is assumed to be $n \geq 3$, we would have $a \leq -2$. Thus $\mu - \frac{n+1}{2} \neq 0$ and so, by (14.31), $p = 0$. This proves the lemma. \square

Hence Proposition 14.13 is proved, and therefore the proof of Theorem 6.7 is completed.

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List of Symbols

$[\pm, \pm, \pm, \pm]$, 72

$*$, Hodge star operator, **86**

$\Lambda^+(N)$, 51

$\Lambda^+(N)_{\text{even}}$, **53**

$\Lambda^+(N)_{\text{BD}}$, **53**

$\Delta = -(dd^* + d^*d)$, Hodge Laplacian, **87**

$\Delta_{\mathbb{R}^{n-1}}$, **5, 93**

$\Delta_{\mathbb{C}^n}$, holomorphic Laplacian, **40**

Ξ , light cone, 14, 118

$\Pi_{\ell, \delta}$, irreducible unitary representation of $O(n+1, 1)$, **20**

Π_{n-1} , projection onto $\text{Ker}(\iota_{\frac{\partial}{\partial x_n}})$, **95**, 99, 110

$\Phi_{u, \delta}^* \equiv \left(\Phi_{u, \delta}^{(i)}\right)^*$, **85**

$\Omega(h, x)$, conformal factor, **2**

$\gamma(\mu, a)$, **6**, 139, 146, 147, 155

$\varepsilon_n(I)$, signature of index set I , **86**

ι , conformal compactification, 15, 121

$\iota_{\lambda}^{(i)}$, map to flat picture, **16**, 113

$\iota_{N_Y(X)}$, **3, 90**

$\iota_{\frac{\partial}{\partial x_n}}$, interior multiplication, **6**, 57, **93**, 96, 99

$[\lambda]$, $O(N)$ -modification rule, **54**

$\lambda \setminus \nu$, skew diagram, **53**

λ/ν , **53**

$\mu^b \equiv \mu^b(i)$, small K -type, 19

$\mu^{\#} \equiv \mu^{\#}(i)$, small K -type, 19

$\xi^{\pm} (\in \Xi)$, **15**

$[\xi^{\pm}] (\in \Xi/\mathbb{R}^{\times} = S^n)$, 15, 21

$[\xi^-]$, north pole in S^n , **15**, 121

$\varpi_{u, \delta}^{(i)}$, conformal representation on i -forms, **2**, 17, **85**

$\pi(\sigma, \lambda)$, principal series, **32**

$\pi(\sigma, \lambda)^*$, **33**

ρ , 18, **32**

ρ_G , 19

$\sigma_{\lambda} := \sigma \boxtimes \mathbb{C}_{\lambda}$, **16, 32**, 35, 39

$\sigma_{\lambda}^* := \sigma^{\vee} \boxtimes \mathbb{C}_{2\rho-\lambda}$, **33**

$\sigma_{\lambda, \alpha}^{(i)}$, representation of P on $\bigwedge^i(\mathbb{C}^n)$, **16, 59**, 61, 81, 109, 127

$\tau_{\nu} \equiv \tau \boxtimes \mathbb{C}_{\nu}$, **34**, 39

$\tau_{\nu, \beta}^{(j)}$, representation of P' on $\bigwedge^j(\mathbb{C}^{n-1})$, **21, 59**, 61, 81, 109

$\vartheta_z = z \frac{d}{dz}$, 74, 155

$\chi_{\pm\pm}$, one-dimensional representation of $O(n+1, 1)$, **16**, 20, 27

χ_{--} , **17**, 22, 113

A , split torus ($\simeq \mathbb{R}$), **15**, 18, 21, 32

$A_{II'}$, matrix component of A_{σ} , **45**, 48, 68, 69, 130

$A_{\#}$, **31**

A_{σ} , vector part of $\widehat{d\pi_{(\sigma, \lambda)^*}}$, **36**, 42, 45

$B^{(k)}$, bilinear map, **48**, 99

$C_{\ell}^{+} (= 2N_{\ell}^{+})$, basis of $\mathfrak{n}_{+}(\mathbb{R})$, **14**

$C_{\ell}^{-} (= N_{\ell}^{-})$, basis of $\mathfrak{n}_{-}(\mathbb{R})$, **14**

$C_{\ell}^{\mu}(t)$, Gegenbauer polynomial, **153**

$\tilde{C}_{\ell}^{\mu}(t)$, renormalized Gegenbauer polynomial, 5, 22, 61, 63, 101, **153**

$\text{Conf}(X)$, **90**, 92, **117**

$\text{Conf}(X; Y)$, **90**, 92, **117**, 118

\mathbb{C}_{λ} , one-dimensional representation of A , **15**

$\mathbb{C}_{2\rho}$, 18, **32**

$\tilde{\mathbb{C}}_{\lambda, \nu} (= \text{Rest}_{x_n=0} \circ \mathcal{D}_{\nu-\lambda}^{\lambda-\frac{n-1}{2}})$, Juhl's operator, **22**

$\mathbb{C}_{\lambda, \nu}^{i, j} (= \mathcal{D}_{\lambda-i, \nu-\lambda}^{i \rightarrow j}): \mathcal{E}^i(\mathbb{R}^n) \longrightarrow \mathcal{E}^j(\mathbb{R}^{n-1})$, (unnormalized) differential symmetry breaking operator, **23**

$\mathbb{C}_{\lambda, \nu}^{i, i-1}$, **23**

$\mathbb{C}_{\lambda, \nu}^{i, i}$, **23**

$\tilde{\mathbb{C}}_{\lambda, \nu}^{i, j} (= \tilde{\mathcal{D}}_{\lambda-i, \nu-j}^{i \rightarrow j})$, normalized differential symmetry breaking operator, **23**

$\tilde{\mathbb{C}}_{\lambda, \nu}^{i, i-2}$, 114

$\tilde{\mathbb{C}}_{n-i, n-i+1}^{i, i-2}$, **24**, 116

- $\tilde{\mathbb{C}}_{\lambda,1}^{n,n-2}$, **24**, 116
 $\tilde{\mathbb{C}}_{\lambda,\nu}^{i,i-1}$, **24**, 25, 110, 114
 $\tilde{\mathbb{C}}_{\lambda,\nu}^{i,i}$, **24**, 25, 114
 $\tilde{\mathbb{C}}_{\lambda,\nu}^{i,i+1}$, **24**, 114
 $\tilde{\mathbb{C}}_{i,i+1}^{i,i+1}$, **24**, 25
 $\tilde{\mathbb{C}}_{\lambda,1}^{0,1}$, **24**
 d , differential, 93
 d^* , codifferential, **5**, 89, 93
 $\mathcal{D}(E)$, Weyl algebra, **31**, 33
 \mathcal{D}_ℓ^μ , **6**, 22, 102
 $\text{Diff}^{\text{const}}$, 34, 98
 $\text{Diff}_{G'}(\mathcal{E}^i(X)_{u,\delta}, \mathcal{E}^j(Y)_{v,\varepsilon})$, **2**
 $\mathcal{D}_{u,a}^{i \rightarrow j} (= \mathbb{C}_{u+i, u+i+a}^{i,j}): \mathcal{E}^i(\mathbb{R}^n) \longrightarrow \mathcal{E}^j(\mathbb{R}^{n-1})$,
(unnormalized) differential symmetry breaking operator, 23
 $\mathcal{D}_{u,a}^{i \rightarrow i-1}$, **6**, 23, 62, 138
 $\mathcal{D}_{u,a}^{i \rightarrow i}$, **7**, 23, 105, 138
 $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow j} (= \tilde{\mathbb{C}}_{u+i, u+i+a}^{i,j}): \mathcal{E}^i(\mathbb{R}^n) \longrightarrow \mathcal{E}^j(\mathbb{R}^{n-1})$,
normalized differential symmetry breaking operator, 23
 $\tilde{\mathcal{D}}_{u,a}^{i \rightarrow i-2}$, **9**
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